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ON PARA-KAHLER-NORDEN PROPERTIES OF THE φ-SASAKI METRIC ON TANGENT BUNDLE

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ABSTRACT. In the present paper, we investigate para-Nordenian properties of the φ-Sasaki metric on the tangent bundle.

Keywords: Horizontal lift and vertical lift, tangent bundle, φ-Sasaki metric, almost para-complex structure, pure metric.

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1. INTRODUCTION

In this field, the notion of almost para-complex structure on a smooth manifold has been studied, in the first papers by Libermann, P. [9], Patterson, E. M.[12] until now, from several different points of view. Moreover, the papers related to it have appeared many times in a rather disperse way, and a survey of further results on para-complex geometry (including para-Kähler geometry) can be found for instance in [2, 3, 5]. Also, other further signifiant developments are due in some recent surveys [1, 8, 13], where some aspects concerning the geometry of para-complex manifolds are presented on the tangent and cotangent bundles. See also [7, 6, 11, 15, 16].

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called φ-Sasaki metric on the tangent bundle TM over a para-Kahler-Norden manifold (M^{2m}, φ, g). This new metric will lead us to interesting results. Afterward we construct almost para-complex Norden structures on tangent bundle equipped with the

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φ-Sasaki metric and investigate necessary and sufficient conditions for these structures to become para-Kähler-Norden, quasi-para-Kähler-Norden. Finally we characterize some properties of almost para-complex Norden structures in context of almost product Riemannian manifolds.

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Let $C^\infty(M)$ (resp. $C^\infty(TM)$) be the ring of real-valued C^∞ functions on M (resp. TM) and $\Im^r_s(M)$ (resp. $\Im^r_s(TM)$) be the module over $C^\infty(M)$ (resp. $C^\infty(TM)$) of C^∞ tensor fields of type (r, s).

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned}\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)},\ a^i \in \mathbb{R}\},\\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ a^i \in \mathbb{R}\},\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

Lemma 2.1. [18] *Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Im^1_0(M)$ we have:*

(1) $[X^H, Y^H]_p = [X, Y]^H_p - (R_x(X, Y)u)^V$,
(2) $[X^H, Y^V]_p = (\nabla_X Y)^V_p$,
(3) $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

An almost product structure φ on a manifold M is a $(1, 1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1, 1)$ on M). The pair (M, φ)

is called an almost product manifold.

A linear connection ∇ on (M, φ) such that $\nabla\varphi = 0$ is said to be an almost product connection. There exists an almost product connection on every almost product manifold[4].

An almost para-complex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [3].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a real $2m$-dimensional differentiable manifold M^{2m} with an almost para-complex structure φ and a Riemannian metric g such that

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{2.3}$$

for all $X, Y \in \Im_0^1(M)$, in this case g is called a pure metric with respect to φ or para-Norden metric (B-metric)[13].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that φ is integrable i.e $\nabla\varphi = 0$ (B-manifold), where ∇ is the Levi-Civita connection of g [13, 16].

A Tachibana operator $\phi_\varphi : \Im_0^2(M) \to \Im_0^3(M)$ applied to the pure metric g is given by

$$\begin{aligned}(\phi_\varphi g)(X, Y, Z) &= (\varphi X)(g(Y, Z)) + X(g(\varphi Y, Z)) + g((L_Y\varphi)X, Z) \\ &\quad + g((L_Z\varphi)X, Y),\end{aligned} \tag{2.4}$$

for all $X, Y, Z \in \Im_0^1(M)$ [17], where L_Y denotes the Lie differentiation with respect to Y.

In an almost para-complex Norden manifold, a para-Norden metric g is called para-holomorphic if

$$(\phi_\varphi g)(X, Y, Z) = 0, \tag{2.5}$$

for all $X, Y, Z \in \Im_0^1(M)$[13].

A para-holomorphic Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that g is a para-holomorphic i.e $\phi_\varphi g = 0$.

In [13], Salimov and his collaborators showed that for an almost para-complex Norden manifold, the condition $\phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$. By virtue of this point of view, para-holomorphic Norden manifolds are similar to para-Kähler-Norden manifolds (For complex version see [8]).

The purity conditions for a tensor field $\omega \in \Im_0^q(M)$ with respect to the para-complex structure φ given by

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, X_2, \cdots, \varphi X_q),$$

for all $X_1, X_2, \cdots, X_q \in \Im_0^1(M)$ [13].

In [17], an operator $\phi_\varphi : \Im_0^q(M) \to \Im_0^{q+1}(M)$ joined with φ and applied to the pure tensor field ω, given by

$$\begin{aligned}(\phi_\varphi\omega)(Y, X_1, \cdots, X_q) &= (\varphi Y)(\omega(X_1, \cdots, X_q)) + Y(\omega(\varphi X_1, \cdots, X_q)) \\ &\quad +\omega((L_{X_1}\varphi)Y, X_2, \cdots, X_q) + \cdots + \omega((X_1, \cdots, (L_{X_q}\varphi)Y),\end{aligned}$$

for all $Y, X_1, X_2, \cdots, X_q \in \Im_0^1(M)$. If $\phi_\varphi\omega$ vanishes, then ω is said to be almost para-holomorphic.

It is well known that if (M^{2m}, φ, g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [13], and

$$\begin{cases} \nabla_Y(\varphi Z) &= \varphi\nabla_Y Z, \\ R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) &= R(Y, Z), \end{cases} \tag{2.6}$$

for all $Y, Z \in \Im_0^1(M)$.

Let (M^{2m}, φ, g) be a non-integrable almost para-complex Norden manifold, if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y, Z) = 0.$$

for all $X, Y, Z \in \Im_0^1(M)$, where σ is the cyclic sum by three arguments, then the triple (M^{2m}, φ, g) is a quasi-para-Kähler-Norden manifold [5, 10]. It is well known that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y, Z) = 0 \Leftrightarrow \underset{X,Y,Z}{\sigma} (\phi_\varphi g)(X, Y, Z) = 0, \tag{2.7}$$

which was proven in [14].

3. φ-Sasaki metric

Definition 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold. On the tangent bundle TM, we define a φ-Sasaki metric noted g_φ by*

$$\begin{aligned} (1)\ g_\varphi(X^H, Y^H)_{(x,u)} &= g_x(X, Y), \\ (2)\ g_\varphi(X^H, Y^V)_{(x,u)} &= 0, \\ (3)\ g_\varphi(X^V, Y^V)_{(x,u)} &= g_x(X, \varphi Y), \end{aligned}$$

where $X, Y \in \Im_0^1(M)$ and $(x,u) \in TM$.

Lemma 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, we have the following*

$$\begin{aligned}
(1)\ X^H g_\varphi(Y^H, Z^H) &= Xg(Y,Z),\\
(2)\ X^V g_\varphi(Y^H, Z^H) &= 0,\\
(3)\ X^H g_\varphi(Y^V, Z^V) &= g_\varphi((\nabla_X Y)^V, Z^V) + g_\varphi(Y^V, (\nabla_X Z)^V),\\
(4)\ X^V g_\varphi(Y^H, Z^H) &= 0,
\end{aligned}$$

for any $X, Y, Z \in \Im_0^1(M)$, where ∇ denote the Levi-Civita connection of (M^{2m}, φ, g).

Theorem 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M, g) (resp (TM, g_φ)), then we have:*

$$\begin{aligned}
(1)\ \ (\widetilde{\nabla}_{X^H} Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V,\\
(2)\ \ (\widetilde{\nabla}_{X^H} Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(\varphi u, Y)X)^H,\\
(3)\ \ (\widetilde{\nabla}_{X^V} Y^H)_{(x,u)} &= \frac{1}{2}(R_x(\varphi u, X)Y)^H,\\
(4)\ \ (\widetilde{\nabla}_{X^V} Y^V)_{(x,u)} &= 0,
\end{aligned}$$

for all vector fields $X, Y \in \Im_0^1(M)$ and $(x,u) \in TM$, where R denote the curvature tensor of (M^{2m}, φ, g).

The proof of Theorem 3.1 follows directly from Kozul formula, Lemma 2.1 and Lemma 3.1.

4. Some almost Para-complex Structures

4.1. We Consider the tensor field $J_\varphi \in \Im_1^1(TM)$ by

$$\begin{cases} J_\varphi X^H &= (\varphi X)^H \\ J_\varphi X^V &= (\varphi X)^V \end{cases} \tag{4.8}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. The couple (TM, J_φ) is an almost para-complex manifold .*

Proof. By virtue of (4.8), we have

$$\begin{cases} J_\varphi^2 X^H = J_\varphi(J_\varphi X^H) = J_\varphi((\varphi X)^H) = (\varphi(\varphi X))^H = (\varphi^2 X)^H = X^H, \\ J_\varphi^2 X^V = J_\varphi(J_\varphi X^V) = J_\varphi((\varphi X)^V) = (\varphi(\varphi X))^V = (\varphi^2 X)^V = X^V, \end{cases}$$

for any $X \in \Im_0^1(M)$, then $J_\varphi^2 = id_{TM}$.

Let $\{E_1, \cdots, E_m, E_{m+1}, \cdots, E_{2m}\}$ be local frame of eigenvectors on M such that

$\varphi E_i = E_i$, $\varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1, m}$.

If $Z = Z_1^i E_i^H + Z_2^i E_i^V$, then

$J_\varphi Z = Z_1^i (\varphi E_i)^H + Z_2^i (\varphi E_i)^V = Z_1^i E_i^H + Z_2^i E_i^V = Z$,

i.e. $TTM^+ = Span(E_1^H, \cdots, E_m^H, E_1^V, \cdots, E_m^V)$,

If $Z = Z_1^{m+i} E_{m+i}^H + Z_2^{m+i} E_{m+i}^V$, then

$J_\varphi Z = Z_1^{m+i} (\varphi E_{m+i})^H + Z_2^{m+i} (\varphi E_{m+i})^V = -Z_{m+1}^i E_{m+i}^H - Z_2^{m+i} E_{m+i}^V = -Z$,

i.e. $TTM^- = Span(E_{m+1}^H, \cdots, E_{2m}^H, E_{m+1}^V, \cdots, E_{2m}^V)$.

Theorem 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*. The triple $(TM, J_\varphi, g_\varphi)$ is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im_0^1(M)$, from (4.8) we have

$$\begin{aligned} (i)\ g_\varphi(J_\varphi X^H, Y^H) &= g_\varphi((\varphi X)^H, Y^H) = g(\varphi X, Y) = g(X, \varphi Y) \\ &= g_\varphi(X^H, (\varphi Y)^H) = g_\varphi(X^H, J_\varphi Y^H), \\ (ii)\ g_\varphi(J_\varphi X^H, Y^V) &= g_\varphi((\varphi X)^H, Y^V) = 0 = g_\varphi(X^H, Y^V) = g_\varphi(X^H, J_\varphi Y^V), \\ (iii)\ g_\varphi(J_\varphi X^V, Y^V) &= g_\varphi((\varphi X)^V, Y^V) = g(\varphi X, \varphi Y) = g(X, Y) \\ &= g(X, \varphi^2 Y) = g_\varphi(X^V, (\varphi Y)^V) = g_\varphi(X^V, J_\varphi Y^V). \end{aligned}$$

Since g is pure with respect to φ, then g_φ is pure with respect to J_φ.

Proposition 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*, then we get*

1. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^H) = 0$,
2. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^H) = 0$,
3. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) = 0$,
4. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^V) = 0$,

5. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^H) = 0$,
6. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^V) = 0$,
7. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^V) = 0$,
8. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^V) = 0$,

for all $X, Y, Z \in \Im^1_0(M)$.

Proof. We calculate Tachibana operator ϕ_{J_φ} applied to the pure metric g_φ. This operator is characterized by (2.4), from Lemma 3.1 we have

$$\begin{aligned}
1.\ (\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^H) &= (J_\varphi X^H) g_\varphi(Y^H, Z^H) - X^H g_\varphi(J_\varphi Y^H, Z^H) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^H, Z^H\big) + g_\varphi\big(Y^H, (L_{Z^H} J_\varphi) X^H\big) \\
&= (\varphi X)^H g_\varphi(Y^H, Z^H) - X^H g_\varphi((\varphi Y)^H, Z^H) \\
&\quad + g_\varphi\big(L_{Y^H} J_\varphi X^H - J_\varphi(L_{Y^H} X^H), Z^H\big) \\
&\quad + g_\varphi\big(Y^H, L_{Z^H} J_\varphi X^H - J_\varphi(L_{Z^H} X^H)\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) \\
&\quad + g_\varphi\big([Y^H, (\varphi X)^H] - J_\varphi [Y^H, X^H], Z^H\big) \\
&\quad + g_\varphi\big(Y^H, [Z^H, (\varphi X)^H] - J_\varphi [Z^H, X^H]\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) + g\big([Y, \varphi X] - \varphi[Y, X], Z\big) \\
&\quad + g\big(Y, [Z, \varphi X] - \varphi[Z, X]\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) + g\big((L_Y \varphi) X, Z\big) \\
&\quad + g\big(Y, (L_Z \varphi) X\big) \\
&= (\phi_\varphi g)(X, Y, Z).
\end{aligned}$$

Since (M^{2m}, φ, g) is a para-Kähler-Norden manifold, then $(\phi_\varphi g)(X, Y, Z) = 0$.

$$\begin{aligned}
2.\ (\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^H) &= (J_\varphi X^V) g_\varphi(Y^H, Z^H) - X^V g_\varphi(J_\varphi Y^H, Z^H)) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^V, Z^H\big) + g_\varphi\big(Y^H, (L_{Z^H} J_\varphi) X^V\big) \\
&= (\varphi X)^V g_\varphi(Y^H, Z^H) - X^V g_\varphi((\varphi Y)^H, Z^H) \\
&\quad + g_\varphi\big([Y^H, (\varphi X)^V] - J_\varphi [Y^H, X^V], Z^H\big) \\
&\quad + g_\varphi\big(Y^H, [Z^H, (\varphi X)^V] - J_\varphi [Z^H, X^V]\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
3.\ (\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) &= (J_\varphi X^H) g_\varphi(Y^V, Z^H) - X^H g_\varphi(J_\varphi Y^V, Z^H) \\
&\quad + g_\varphi\big((L_{Y^V} J_\varphi) X^H, Z^H\big) + g_\varphi\big(Y^V, (L_{Z^H} J_\varphi) X^H\big) \\
&= g_\varphi\big([Y^V, (\varphi X)^H] - J_\varphi [Y^V, X^H], Z^H\big) \\
&\quad + g_\varphi\big(Y^V, [Z^H, (\varphi X)^H] - J_\varphi [Z^H, X^H]\big) \\
&= g_\varphi\big(Y^V, (-R(Z, \varphi X)u)^V + (\varphi R(Z, X)u)^V\big) \\
&= -g\big(R(Z, \varphi X)u, \varphi Y\big) + g\big(\varphi R(Z, X)u, \varphi Y\big).
\end{aligned}$$

Since the Riemann curvature R of a para-Kähler-Norden manifold is pure, then

$$\begin{aligned}
(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) &= -g\big(R(Z, X)u, Y\big) + g\big(R(Z, X)u, Y\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
4.\ (\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^V) &= (J_\varphi X^H) g_\varphi(Y^H, Z^V) - X^H g_\varphi(J_\varphi Y^H, Z^V) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^H, Z^V\big) + g_\varphi\big(Y^H, (L_{Z^V} J_\varphi) X^H\big) \\
&= g_\varphi\big([Y^H, (\varphi X)^H] - J_\varphi [Y^H, X^H], Z^V\big) \\
&\quad + g_\varphi\big(Y^H, [Z^V, (\varphi X)^H] - J_\varphi [Z^V, X^H]\big) \\
&= g_\varphi\big((-R(Y, \varphi X)u)^V + (\varphi R(Y, X)u)^V, Z^V\big) \\
&= -g(R(Y, \varphi X)u, \varphi Z) + g(\varphi R(Y, X)u, \varphi Z) \\
&= -g(R(Y, X)u, Z) + g(R(Y, X)u, Z) \\
&= 0.
\end{aligned}$$

The other formulas are obtained by a similar calculation.

Therefore, we have the following result.

Theorem 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*,then the triple $(TM, J_\varphi, g_\varphi)$ is a para-Kähler-Norden manifold.*

Corollary 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*, then the triple $(TM, J_\varphi, g_\varphi)$ is a quasi-para-Kähler-Norden manifold.*

4.2. We Consider the tensor field $P_\varphi \in \Im_1^1(TM)$ defined by:

$$\begin{cases} P_\varphi X^H &= -(\varphi X)^H \\ P_\varphi X^V &= -(\varphi X)^V \end{cases} \tag{4.9}$$

for all $X \in \Im_0^1(M)$, satisfies the following:

1. $P\varphi = -J\varphi$.
2. g_φ is pure with respect to P_φ.
3. $\phi_{P_\varphi} g_\varphi = \phi_{J_\varphi} g_\varphi$.

Therefore we have the following results.

Theorem 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure P_φ defined by* (4.9)*, then the triple $(TM, P_\varphi, g_\varphi)$ is a para-Kähler-Norden manifold.*

4.3. We Consider the tensor field $Q_\varphi \in \Im_1^1(TM)$ defined by:

$$\begin{cases} Q_\varphi X^H &= (\varphi X)^V \\ Q_\varphi X^V &= (\varphi X)^H \end{cases} \tag{4.10}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. The couple (TM, Q_φ) is an almost para-complex manifold .*

Proof. By virtue of (4.10), we have

$$\begin{cases} Q_\varphi^2 X^H = Q_\varphi(Q_\varphi X^H) = Q_\varphi((\varphi X)^V) = (\varphi(\varphi X))^H = (\varphi^2 X)^H = X^H, \\ Q_\varphi^2 X^V = Q_\varphi(Q_\varphi X^V) = Q_\varphi((\varphi X)^H) = (\varphi(\varphi X))^V = (\varphi^2 X)^V = X^V, \end{cases}$$

for any $X \in \Im_0^1(M)$, then $Q_\varphi^2 = id_{TM}$.

Let $\{E_1, \cdots, E_m, E_{m+1}, \cdots, E_{2m}\}$ be local frame of eigenvectors on M such that $\varphi E_i = E_i$, $\varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1, m}$, then

$$TTM^+ = Span(E_1^H + E_1^V, \cdots, E_m^H + E_m^V, E_{m+1}^H - E_{m+1}^V, \cdots, E_{2m}^H - E_{2m}^V),$$

$$TTM^- = Span(E_1^H - E_1^V, \cdots, E_m^H - E_m^V, E_{m+1}^H + E_{m+1}^V, \cdots, E_{2m}^H + E_{2m}^V).$$

Theorem 4.4. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure Q_φ defined by (4.10). The φ-Sasaki metric is never pure with respect to Q_φ i.e The triple $(TM, Q_\varphi, g_\varphi)$ is never an almost para-complex Norden manifold.*

4.4. We Consider the tensor field $F_\varphi \in \Im_1^1(TM)$ by

$$\begin{cases} F_\varphi X^H &= -(\varphi X)^H \\ F_\varphi X^V &= (\varphi X)^V \end{cases} \tag{4.11}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. The couple (TM, F_φ) is an almost para-complex manifold .*

Theorem 4.5. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by (4.11). The triple $(TM, F_\varphi, g_\varphi)$ is an almost para-complex Norden manifold.*

Proof. With the same steps in the proof of Theorem 4.1, we get the results.

Proposition 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by (4.11), then we get*

1. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^H, Z^H) = 0,$
2. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^H, Z^H) = 0,$
3. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^V, Z^H) = 2g\big(R(X,Z)Y, u\big),$
4. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^H, Z^V) = 2g\big(R(X,Y)Z, u\big),$
5. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^V, Z^H) = 0,$
6. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^H, Z^V) = 0,$
7. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^V, Z^V) = 0,$
8. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^V, Z^V) = 0,$

for all $X, Y, Z \in \Im_0^1(M)$, where R denote the curvature tensor of (M, g).

Proof. We calculate Tachibana operator ϕ_{F_φ} applied to the pure metric g_φ. With the same steps in the proof of Proposition 4.1, we get the results.

Theorem 4.6. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by* (4.11). *The triple $(TM, F_\varphi, g_\varphi)$ is a para-Kähler-Norden manifold if and only if M is flat.*

Proof. For all $X, Y, Z \in \Im^1_0(M)$ and $h, k, l \in \{H, V\}$

$$(\phi_{F_\varphi} g_\varphi))(X^h, Y^k, Z^l) = 0 \quad \Leftrightarrow \quad \begin{cases} g(R(X,Z)Y, u) & = 0 \\ g(R(X,Y)Z, u) & = 0 \end{cases}$$

$$\Leftrightarrow \quad R \ = 0.$$

Theorem 4.7. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by* (4.11). *The triple $(TM, F_\varphi, g_\varphi)$ is a quasi-para-Kähler-Norden manifold.*

Proof. For all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(TM)$,

$$\underset{\widetilde{X},\widetilde{Y},\widetilde{Z}}{\sigma}(\phi_{J_\varphi} g_\varphi)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = (\phi_{J_\varphi} g_\varphi)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) + (\phi_{J_\varphi} g_\varphi)(\widetilde{Y}, \widetilde{Z}, \widetilde{X}) + (\phi_{J_\varphi} g_\varphi)(\widetilde{Z}, \widetilde{X}, \widetilde{Y})$$

By virtue of Proposition 4.1 we have

1. $\underset{X^H,Y^H,Z^H}{\sigma}(\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^H) \ = \ 0,$
2. $\underset{X^V,Y^H,Z^H}{\sigma}(\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^H) \ = \ 2g(R(Y,Z)X, u) + 2g(R(Z,Y)X, u) = 0,$
3. $\underset{X^V,Y^V,Z^H}{\sigma}(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^H) \ = \ 0,$
4. $\underset{X^V,Y^V,Z^V}{\sigma}(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^V) \ = \ 0,$

then, $(TM, J_\varphi, g_\varphi)$ is a quasi-para-Kähler-Norden manifold.

4.5. We Consider the tensor field $K_\varphi \in \Im^1_1(TM)$ defined by:

$$\begin{cases} K_\varphi X^H & = (\varphi X)^H \\ K_\varphi X^V & = -(\varphi X)^V \end{cases} \tag{4.12}$$

for all $X \in \Im^1_0(M)$, satisfies the following:

1. $K\varphi = -F\varphi$.
2. g_φ is pure with respect to K_φ.
3. $\phi_{K_\varphi} g_\varphi = -\phi_{F_\varphi} g_\varphi$.

Therefore we have the following results.

Theorem 4.8. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure K_φ defined by* (4.12)*, then we have*

1. *The triple (TM,K_φ,g_φ) is is a quasi-para-Kähler-Norden manifold.*
2. *The triple (TM,K_φ,g_φ) is a para-Kähler-Norden manifold if and only if M is flat.*

4.6. Now consider the almost product structure F_φ defined by (4.11). We define a tensor field S of type $(1,2)$ and linear connection $\widehat{\nabla}$ on TM by,

$$S(\widetilde{X},\widetilde{Y}) = \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi\widetilde{Y}}F_\varphi)\widetilde{X}+F_\varphi\big((\widetilde{\nabla}_{\widetilde{Y}}F_\varphi)\widetilde{X}\big)-F_\varphi\big((\widetilde{\nabla}_{\widetilde{X}}F_\varphi)\widetilde{Y}\big)\big]. \tag{4.13}$$

$$\widehat{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}-S(\widetilde{X},\widetilde{Y}). \tag{4.14}$$

for all $\widetilde{X},\widetilde{Y}\in\Im^1_0(TM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection of (TM,g_φ) given by Theorem 3.1. $\widehat{\nabla}$ is an almost product connection on TM (see [4, p.151] for more details).

Lemma 4.4. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by* (4.11)*. Then tensor field S is as follows,*

(1) $S(X^H,Y^H)=-\frac{1}{2}(R(X,Y)u)^V$,
(2) $S(X^H,Y^V)=\frac{1}{2}(R(\varphi u,Y)X)^H$,
(3) $S(X^V,Y^H)=-(R(\varphi u,X)Y)^H$,
(4) $S(X^V,Y^V)=0$,

for all $X,Y\in\Im^1_0(M)$.

Proof. (1) Using (4.11) and (4.13), we have

$$\begin{aligned}
S(X^H,Y^H) &= \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi Y^H}F_\varphi)X^H+F_\varphi\big((\widetilde{\nabla}_{Y^H}F_\varphi)X^H\big)-F_\varphi\big((\widetilde{\nabla}_{X^H}F_\varphi)Y^H\big)\big]\\
&= \frac{1}{2}\big[\widetilde{\nabla}_{(\varphi Y)^H}(\varphi X)^H+F_\varphi(\widetilde{\nabla}_{(\varphi Y)^H}X^H)-F_\varphi\big(\widetilde{\nabla}_{Y^H}(\varphi X)^H\big)\\
&\quad -\widetilde{\nabla}_{Y^H}X^H+F_\varphi\big(\widetilde{\nabla}_{X^H}(\varphi Y)^H\big)+\widetilde{\nabla}_{X^H}Y^H\big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\big[(\nabla_{\varphi Y}\varphi X)^H-\frac{1}{2}(R(\varphi Y,\varphi X)u)^V-(\varphi\nabla_{\varphi Y}X)^H\\
&\quad -\frac{1}{2}(\varphi R(\varphi Y,X)u)^V+(\varphi\nabla_Y\varphi X)^H+\frac{1}{2}(\varphi R(Y,\varphi X)u)^V\\
&\quad -(\nabla_Y X)^H+\frac{1}{2}(R(Y,X)u)^V-(\varphi\nabla_X\varphi Y)^H\\
&\quad -\frac{1}{2}(\varphi R(X,\varphi Y)u)^V+(\nabla_X Y)^H-\frac{1}{2}(R(X,Y)u)^V\big].
\end{aligned}$$

Using (2.6) we have

$$S(X^H, Y^H) = -\frac{1}{2}(R(X,Y)u)^V.$$

(2) By a similar calculation to (1), we have

$$\begin{aligned} S(X^H, Y^V) &= \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi Y^V} F_\varphi)X^H + F_\varphi\big((\widetilde{\nabla}_{Y^V} F_\varphi)X^H\big) - F_\varphi\big((\widetilde{\nabla}_{X^H} F_\varphi)Y^V\big)\big] \\ &= \frac{1}{2}\big[-\widetilde{\nabla}_{(\varphi Y)^V}(\varphi X)^H - F_\varphi(\widetilde{\nabla}_{(\varphi Y)^V} X^H) - F_\varphi\big(\widetilde{\nabla}_{Y^V}(\varphi X)^H\big) \\ &\quad -\widetilde{\nabla}_{Y^V} X^H - F_\varphi\big(\widetilde{\nabla}_{X^H}(\varphi Y)^V\big) + \widetilde{\nabla}_{X^H} Y^V\big] \\ &= \frac{1}{2}\big[-\frac{1}{2}(R(\varphi u, \varphi Y)\varphi X)^H + \frac{1}{2}(\varphi R(\varphi u, \varphi Y)X)^H \\ &\quad +\frac{1}{2}(\varphi R(\varphi u, Y)\varphi X)^H - \frac{1}{2}(R(\varphi u, Y)X)^H \\ &\quad -(\varphi\nabla_X \varphi Y)^V + \frac{1}{2}(\varphi R(\varphi u, \varphi Y)X)^H \\ &\quad +(\nabla_X Y)^V + \frac{1}{2}(R(\varphi u, Y)X)^H\big]. \end{aligned}$$

Using (2.6) we get

$$S(X^H, Y^V) = \frac{1}{2}(R(\varphi u, Y)X)^H.$$

The other formulas are obtained by a similar calculation.

Theorem 4.9. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by* (4.11)*. Then the almost product connection $\widehat{\nabla}$ defined by* (4.14) *is as follows,*

$$\begin{aligned} &(1)\ \widehat{\nabla}_{X^H} Y^H = (\nabla_X Y)^H, \\ &(2)\ \widehat{\nabla}_{X^H} Y^V = (\nabla_X Y)^V, \\ &(3)\ \widehat{\nabla}_{X^V} Y^H = \frac{3}{2}(R(\varphi u, X)Y)^H, \\ &(4)\ \widehat{\nabla}_{X^V} Y^V = 0, \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$.

Proof. The proof of Theorem 4.9 follows directly from Theorem 3.1, Lemma 4.4 and formula (4.14).

Lemma 4.5. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by*

(4.11) *and* $\widehat{T}$ *denote the torsion tensor of* $\widehat{\nabla}$, *then we have:*

$$\begin{aligned}
(1)\ \widehat{T}(X^H, Y^H) &= (R(X,Y)u)^V, \\
(2)\ \widehat{T}(X^H, Y^V) &= -\frac{3}{2}(R(\varphi u, Y)X)^H, \\
(3)\ \widehat{T}(X^V, Y^H) &= \frac{3}{2}(R(\varphi u, X)Y)^H, \\
(4)\ \widehat{T}(X^V, Y^V) &= 0,
\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$.

Proof. The proof of Lemma 4.5 follows directly from Lemma 4.4 and formula

$$\begin{aligned}
\widehat{T}(\widetilde{X}, \widetilde{Y}) &= \widehat{\nabla}_{\widetilde{X}}\widetilde{Y} - \widehat{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X}, \widetilde{Y}] \\
&= S(\widetilde{Y}, \widetilde{X}) - S(\widetilde{X}, \widetilde{Y})
\end{aligned}$$

for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(TM)$.

From Lemma 4.5 we obtain

Theorem 4.10. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold,* (TM, g_φ) *be its tangent bundle equipped with the* φ*-Sasaki metric and the almost product structure* F_φ *defined by* (4.11), *then* $\widehat{\nabla}$ *is symmetric if and only if* M *is flat.*

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