**International Journal of Maps in Mathematics**

*Volume 4, Issue 2, 2021, Pages:**93-106***

ISSN: 2636-7467 (Online)

**www.journalmim.com**

# A CONSTRUCTION OF VERY TRUE OPERATOR ON SHEFFER STROKE MTL-ALGEBRAS

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Abstract. In this paper, we introduce Sheffer stroke very true operator on MTL-algebras. We handle some fundamental properties of this operator. We obtain some equalities and inequalities which are used in our construction. Moreover, we give some relations among very true operator, supremum and infimum relations. Finally, we construct bridges among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them.

**Keywords**: Sheffer stroke MTL-algebra, Very True Operator, Reduction, BL-algebra, MV-algebra, Gödel algebra.

**2010 Mathematics Subject Classification**: 03F50, 06F99.

## 1. Introduction

When a structure is established as a mathematical model, we must firstly throw off redundant statements. For this aim, we venture to give equivalent statements as possible as with the least number of axioms or the least number of operations and so on. For instance, Tarski achieved to explain Abelian groups with the least number of axioms from the point of divisor operator. [19]

The concept of monoidal t-norm-based logic (shortly, MTL) is given by Godo and Esteva [8]. Montogna and Jenei show that MTL corresponds to the logic of all left continuous t-norms and their residua [11]. In accordance wtih these studies, MTL-algebras are defined as a counterpart of this logical system [8]. In recent times, the structure of MTL-algebras

*Received:2021.01.20* *Revised:2021.03.09* *Accepted:2021.03.19*

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has been supported with important structural works [13, 20]. These works get a constructive effect on its algebraic structure. For instance, Vetterlein demonstrate that MTL-algebras correspond to the positive cone of a partially ordered group [20]. Moreover, he confirm that this algebra is a commutative, bounded, integral and pre-linear residuated lattice [13]. And, MTL-algebras are the basis residuated structures having all algebras induced by their residua and continuous t-norms. So, MTL-algebras have an important position in different structures which are related with fuzzy logic [21].

Oner and Senturk introduced Sheffer stroke basic algebras [14]. Sheffer stroke basic algebras play an important role in great numbers of logics as many-valued Łukasiewicz logics, non-classical logics, fuzzy logics and etc. This reduction topic is studied in recent times such as [15]. In harmony with these logical roles, Senturk gives a reduction of MTL-algebras by means of only Sheffer stroke operation which is called Sheffer stroke MTL-algebras [18].

The notion of "very true" was firstly established by Hájek giving an answer to the question "whether any natural axiomatization is possible and how far can even this sort of fuzzy logic be captured by standard methods of mathematical logic?" [10]. To put in a different way, very true operator is used to reduce the number of possible logical values in many-valued logic. After this operator was effectively used in particular tasks in various fields of mathematics [9, 5, 1, 23], this operator has been implemented to other logical algebras such as effect algebras [6], commutative basic algebras [3], equality algebras [22], R$\ell$-monoids [16], MV-algebras [12] and so on.

In this paper, we give some fundamental concepts which are needed for our construction in Section 2. In Section 3, we introduce Sheffer stroke very true operator on Sheffer stroke MTL-algebras. We handle some fundamental properties of this operator. We obtain some equalities and inequalities. We give some relations among very true operator, supremum and infimum. Then, we engaging links among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them. In Section 4, we briefly mention what we do during this work.

## 2. Preliminaries

The basic definitions, lemmas, theorems and etc. which are used throughout the paper are given in this section.

The fundamental concepts in this chapter are taken from [17] and [2].

**Definition 2.1.** *If the binary operations $\vee$ and $\wedge$ satisfy the following conditions on the non-empty set $L$:*

$(L_1)$ $k \wedge l = l \wedge k$ *and* $k \vee l = l \vee k$,

$(L_2)$ $k \wedge (l \wedge m) = (k \wedge l) \wedge m$ *and* $k \vee (l \vee m) = (k \vee l) \vee m$,

$(L_3)$ $k \wedge k = k$ *and* $k \vee k = k$,

$(L_4)$ $k \wedge (k \vee l) = k$ *and* $k \vee (k \wedge l) = k$

*then* $\mathfrak{L} = (L; \wedge, \vee)$ *is called a lattice.*

**Definition 2.2.** *An algebraic structure* $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ *is called bounded lattice if it satisfies the following properties:*

(i) *for each* $k \in L$, $k \wedge 1 = k$ *and* $k \vee 1 = 1$,

(ii) *for each* $k \in L$, $k \wedge 0 = 0$ *and* $k \vee 0 = k$.

*The elements* 1 *is called the greatest element and* 0 *is called called the least element of the lattice.*

**Definition 2.3.** *Let the structure* $\mathcal{L} = (L; \vee, \wedge)$ *be a lattice. A mapping* $k \mapsto k^{\perp}$ *is said to be an antitone involution if it verifies the following conditions:*

(i) $k^{\perp\perp} = k$ *(involution),*

(ii) $k \leq l$ *implies* $l^{\perp} \leq k^{\perp}$ *(antitone).*

**Definition 2.4.** *Let* $\mathcal{L}$ *be a bounded lattice with an antitone involution. If the below conditions*

$$k \vee k^{\perp} = 1 \quad \textit{and} \quad k \wedge k^{\perp} = 0,$$

*are satisfied then* $k^{\perp}$ *is called the complement of* $k$ *and the lattice* $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ *is also an ortholattice.*

**Lemma 2.1.** *Let* $\mathcal{L} = (L; \vee, \wedge, ^{\perp})$ *be a lattice which verifies the antitone involution condition. Then the De Morgan laws*

$$k^{\perp} \wedge l^{\perp} = (k \vee l)^{\perp} \textit{ and } k^{\perp} \vee l^{\perp} = (k \wedge l)^{\perp}$$

*are satisfied.*

**Definition 2.5.** [4] *Let* $\mathcal{G} = (G, |)$ *be a groupoid. If the following conditions are satisfied, then the operation* $| : G \times G \rightarrow G$ *is called a Sheffer stroke operation.*

$(S1)$ $g_1|g_2 = g_2|g_1$,

$(S2)$ $(g_1|g_1)|(g_1|g_2) = g_1$,

$(S3)$ $g_1|((g_2|g_3)|(g_2|g_3)) = ((g_1|g_2)|(g_1|g_2))|g_3$,

$(S4)$ $(g_1|((g_1|g_1)|(g_1|g_1)))|(g_1|((g_1|g_1)|(g_2|g_2))) = g_1$.

*If also the following identity*

$(S5)$ $g_2|(g_1|(g_1|g_1)) = g_2|g_2$,

*is satisfied, then it is said to be an ortho-Sheffer stroke operation.*

**Lemma 2.2.** [4] *Let $\mathcal{G} = (G, |)$ be a groupoid with Sheffer stroke operation. Then the following equalities are verified for each $g_1, g_2, g_3 \in G$:*

*(i)* $(g_1|g_2)|(g_1|(g_2|g_3)) = g_1$,

*(ii)* $(g_1|g_1)|g_2 = g_2|(g_1|g_2)$,

*(iii)* $g_1|((g_2|g_2)|g_1) = g_1|g_2$.

**Lemma 2.3.** [4] *Let $\mathcal{G} = (G, |)$ be a groupoid. The binary relation $\leq$ defined on $G$ as below*

$$g_1 \leq g_2 \text{ if and only if } g_1|g_2 = g_1|g_1$$

*is a partial order on $G$.*

**Lemma 2.4.** [4] *Let $|$ be a Sheffer stroke operation on $G$ and $\leq$ order relation of $\mathcal{G}$. Then, the following equalities:*

(i) $g_1 \leq g_2$ *if and only if* $g_2|g_2 \leq g_1|g_1$,

(ii) $g_1|(g_2|(g_1|g_1)) = g_1|g_1$ *is the identity of* $\mathcal{G}$,

(iii) $g_1 \leq g_2$ *implies* $g_2|g_3 \leq g_1|g_3$, *for all* $g_3 \in G$,

(iv) $g_3 \leq g_1$ *and* $g_3 \leq g_2$ *imply* $g_1|g_2 \leq g_3|g_3$

*are verified.*

**Lemma 2.5.** [14] *Let $\mathfrak{G} = (G; |)$ be a Sheffer stroke basic algebra with the constant element 1. Then, the following identities:*

(i) $g_1|(g_1|g_1) = 1$,

(ii) $g_1|(1|1) = 1$,

(iii) $1|(g_1|g_1) = g_1$,

(iv) $((g_1|(g_2|g_2))|(g_2|g_2))|(g_2|g_2) = g_1|(g_2|g_2)$,

(v) $(g_2|(g_1|(g_2|g_2)))|(g_1|(g_2|g_2)) = 1$

*are verified.*

**Definition 2.6.** [21] *Let $X$ be a non-empty set, the operations $\vee$, $\wedge$, $\rightarrow$ and $\circledast$ be binary operations on $X$ and the elements 0 and 1 be algebraic constant of $X$. If the following* *conditions:*

$(MTL_1)$ $(X; \wedge, \vee, 0, 1)$ *is a bounded lattice,*

$(MTL_2)$ $(X; \circledast, 0, 1)$ *is a commutative monoid,*

$(MTL_3)$ $x \leq y \rightarrow z$ *if and only if* $x \circledast y \leq z$*,*

$(MTL_4)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$

*are satisfied for each* $x, y, z \in X$*, then the algebraic structure* $\mathcal{X} = (X; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ *is called an MTL-algebra.*

**Definition 2.7.** [21] *Let* $\mathcal{X} = (X; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ *be an MTL-algebra. Then* $\mathcal{X}$ *is called*

$(i)$ *a BL-algebra if* $x \wedge y = x \circledast (x \rightarrow y)$ *for each* $x, y \in X$*,*

$(ii)$ *an MV-algebra if* $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ *for each* $x, y \in X$*,*

$(iii)$ *a Gödel algebra if* $x \circledast x = x$ *for each* $x \in X$*.*

**Theorem 2.1.** [18] *Let* $\mathcal{X} = (X; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ *an* $MTL$*-algebra. If the operations are defined as:*

$x_1 \wedge x_2 := (((x_2|x_2)|x_1)|x_1)|(((x_2|x_2)|x_1)|x_1)$

$x_1 \vee x_2 := (x_1|(x_2|x_2))|(x_2|x_2)$

$x_1 \circledast x_2 := (x_1|x_2)|(x_1|x_2)$

$x_1 \rightarrow x_2 := x_1|(x_2|x_2)$

*for each* $x_1, x_2 \in X$*, then* $\mathcal{X} = (X; |)$ *is a Sheffer stroke reduction of* $MTL$*-algebra.*

**Corollary 2.1.** [18] *Let* $\mathcal{X} = (X; |)$ *is a Sheffer stroke reduction of MTL-algebra. Then, it is also a Sheffer stroke basic algebra.*

During this paper, Sheffer stroke reduction of MTL-algebras are shortly called Sheffer stroke MTL-algebras.

## 3. A Construction of Very True Operator On Sheffer Stroke MTL-Algebras

In this part of the paper, we construct Sheffer stroke very true operator on Sheffer stroke MTL-algebras. We examine some fundamental properties of this operator. We attain some equalities and inequalities. Moreover, we give some relations among very true operator, supremum and infimum. On the other hand, we build links among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them.

**Definition 3.1.** *Let* $\mathcal{M} = (M; |)$ *be a Sheffer stroke* $MTL$*-algebra. If the following conditions:*

$(SV_{SM}1)$ $\vartheta(1) = 1$

$(SV_{SM}2)$ $\vartheta(m) \leq m$

$(SV_{SM}3)$ $\vartheta(m|(n|n)) \leq \vartheta(m)|(\vartheta(n)|\vartheta(n))$

$(SV_{SM}4)$ $\vartheta(m) \leq \vartheta(\vartheta(m))$

$(SV_{SM}5)$ $(\vartheta(m|(n|n))|(\vartheta(n|(m|m))|\vartheta(n|(m|m))))|(\vartheta(n|(m|m))|\vartheta(n|(m|m))) = 1$

*are satisfied for each $m, n \in M$, then the mapping $\vartheta : M \to M$ is called a Sheffer stroke very true operator.*

**Example 3.1.** *Let $M = \{0, k, l, m, n, 1\}$. The relations of elements in $M$ are given as Figure 1 and the operation $|$ on this structure is defined as the Table 1.*

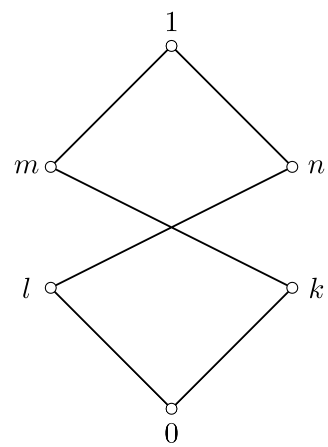


**Figure 1.** Hasse Diagram of $M$

| $\vert$ | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $l$ | 1 | $m$ | 1 | 1 | $m$ | $m$ |
| $k$ | 1 | 1 | $n$ | $n$ | 1 | $n$ |
| $m$ | 1 | 1 | $n$ | $l$ | 1 | $l$ |
| $n$ | 1 | $m$ | 1 | 1 | $k$ | $k$ |
| 1 | 1 | $m$ | $n$ | $l$ | $k$ | 0 |

**Table 1.** $|-$operation on $M$

*If the binary operations $\wedge, \vee, \circledast$ and $\to$ are defined as Theorem 2.1, then we have the following Cayley tables for these operations.*

| $\wedge$ | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $l$ | 0 | $l$ | 0 | 0 | $l$ | $l$ |
| $k$ | 0 | 0 | $k$ | $k$ | 0 | $k$ |
| $m$ | 0 | 0 | $k$ | $m$ | 0 | $m$ |
| $n$ | 0 | $l$ | 0 | 0 | $n$ | $n$ |
| 1 | 0 | $l$ | $k$ | $m$ | $n$ | 1 |

**Table 2.** $\wedge-$operation on $M$

*and*

| $\vee$ | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
| $l$ | $l$ | $l$ | 1 | 1 | $n$ | 1 |
| $k$ | $k$ | 1 | $k$ | $m$ | 1 | 1 |
| $m$ | $m$ | 1 | $m$ | $m$ | 1 | 1 |
| $n$ | $n$ | $n$ | 1 | 1 | $n$ | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 3.** $\vee-$operation on $M$

| $\circledast$ | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $l$ | 0 | $l$ | 0 | 0 | $l$ | $l$ |
| $k$ | 0 | 0 | $k$ | $k$ | 0 | $k$ |
| $m$ | 0 | 0 | $n$ | $m$ | 0 | $m$ |
| $n$ | 0 | $l$ | 0 | $m$ | $n$ | $n$ |
| 1 | 0 | $l$ | $k$ | $m$ | $n$ | 1 |

**Table 4.** $\circledast-$operation on $M$

*and*

| $\rightarrow$ | 0 | $l$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $l$ | $m$ | 1 | $m$ | $m$ | 1 | 1 |
| $k$ | $n$ | $m$ | 1 | 1 | $n$ | 1 |
| $m$ | $l$ | $l$ | 1 | 1 | $n$ | 1 |
| $n$ | $k$ | 1 | $k$ | $m$ | 1 | 1 |
| 1 | 0 | $l$ | $k$ | $m$ | $n$ | 1 |

**Table 5.** $\rightarrow-$operation on $M$

*So, the algebraic structure $\mathcal{M} = (M; |)$ is a Sheffer stroke $MTL$-algebra. If the operation $\vartheta : M \rightarrow M$ is defined by*

$$\vartheta(u) := \begin{cases} 0, & u = 0, \\ 1, & u = 1, \\ n, & u \in \{l, n\}, \\ m, & x \in \{k, m\} \end{cases}$$

*then, this mapping is a Sheffer stroke very true operator on $M$.*

**Proposition 3.1.** *Assume that the mapping $\vartheta : M \rightarrow M$ be a Sheffer stroke very true operator. Then, the following statements*
*(i) $\vartheta(0) = 0$,*
*(ii) $m = 1$ if and only if $\vartheta(m) = 1$,*
*(iii) $\vartheta$ is increasing,*
*(iv) $\vartheta(m|m) \leq \vartheta(m)|\vartheta(m)$*
*hold for each $m, n, k \in L$.*

**Proof.** *(i)* By $(SV_{SM}2)$, we get $\vartheta(0) \leq 0$. Moreover, we have $m \leq \vartheta(m)$ for each $m \in M$. So, we obtain $\vartheta(0) = 0$.
*(ii)* ($\Rightarrow$:) It is clear from $(SV_{SM}1)$.
($\Leftarrow$:) Assume that $\vartheta(m) = 1$. Since $\vartheta(m) = 1 \leq m \leq 1$, we get $m = 1$.
*(iii)* Assume that $m \leq n$. Then, we have $m|(n|n) = 1$. By the help of $(SV_{SM}1)$ and $(SV_{SM}3)$, we get $\vartheta(m|(n|n)) = \vartheta(1) = 1 \leq \vartheta(m)|(\vartheta(n)|\vartheta(n)) \leq 1$. We obtain $\vartheta(m)|(\vartheta(n)|\vartheta(n)) = 1$. So, we conclude that $\vartheta(m) \leq \vartheta(n)$, i.e., the mapping $\vartheta$ is increasing.

$(iv)$ Let $m$ be any element of $M$. Then, we have

$$\begin{aligned}\vartheta(m|m) &= \vartheta(m|1)\\ &= \vartheta(m|(0|0))\\ &\leq \vartheta(m)|(\vartheta(0)|\vartheta(0))\\ &= \vartheta(m)|(0|0)\\ &= \vartheta(m)|1\\ &= \vartheta(m)|\vartheta(m).\end{aligned}$$

So, the inequality $\vartheta(m|m) \leq \vartheta(m)|\vartheta(m)$ is verified for each $m \in M$.

**Lemma 3.1.** *Let $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then, the equality $\vartheta(m) = \vartheta^2(m)$ is verified for each $m \in M$.*

**Proof.** Let $m$ be any element of $M$. By using Proposition 3.1 $(iii)$ and $(SV_{SM}1)$, we obtain $\vartheta(\vartheta(m)) \leq \vartheta(m)$. From $(SV_{SM}4)$, we have $\vartheta(m) \leq \vartheta(\vartheta(m))$. Hence, we obtain $\vartheta(m) = \vartheta(\vartheta(m))$ for each $m \in M$.

**Lemma 3.2.** *The following inequalities*

$$(\vartheta(m)|\vartheta(n))|(\vartheta(m)|\vartheta(n)) \leq (m|n)|(m|n) \leq \vartheta(m|n)|\vartheta(m|n)$$

*hold for each $m, n \in L$.*

**Proof.** Let $m$ and $n$ be any elements of $M$. By using $(SV_{SM}2)$, we get $\vartheta(m) \leq m$ and $\vartheta(n) \leq n$. From Lemma 2.4 $(i)$, we have $m|n \leq \vartheta(m)|\vartheta(n)$. If we use again the same step for the last equation, we get the following inequality:

$$(\vartheta(m)|\vartheta(n))|(\vartheta(m)|\vartheta(n)) \leq (m|n)|(m|n). \tag{3.1}$$

By $(SV_{SM}2)$, we have $\vartheta(m|n) \leq m|n$. Similarly, we obtain

$$(m|n)|(m|n) \leq \vartheta(m|n)|\vartheta(m|n). \tag{3.2}$$

From Inequalities (3.1) and (3.2), we attain our assumption.

**Lemma 3.3.** *The following inequalities*

$(i)$ $\vartheta(m|m) \leq \vartheta(m|n)$,

$(ii)$ $\vartheta(m|n)|\vartheta(m|n) \leq \vartheta(m)$, *(iii)* $\vartheta(m) \leq \vartheta((m|n)|n)$

*hold for each* $m, n \in L$.

**Proof.** *(i)* Let $m$ and $n$ be any two elements of $M$. We have $m \leq 1$ and $n \leq 1$. Then,

$$\begin{aligned} n \leq 1 \quad &\Rightarrow \quad 1|m \leq n|m, && \text{(By Lemma 2.4 } (iii)) \\ &\Rightarrow \quad m|m \leq n|m, && \text{(By Lemma 2.5 and Corollary 2.1)} \\ &\Rightarrow \quad \vartheta(m|m) \leq \vartheta(m|n). && \text{(By Proposition 3.1 } (iii)) \end{aligned}$$

*(ii)* We have the inequality $m|m \leq n|m$ from Lemma 3.3 *(i)*. By the help of Lemma 2.4 *(i)* and Definition 2.5 $(S2)$, we get $(n|m)|(n|m) \leq m$. By increasing property of $\vartheta$ mapping, we conclude that $\vartheta((n|m)|(n|m)) \leq \vartheta(m)$ for each $m, n \in M$.

*(iii)* We have $n \leq 1$ for each $n \in M$. We obtain $m \leq (m|n)|n$ by using Lemma 2.4 *(iii)*, Lemma 2.5 *(iii)* and Lemma 2.2, respectively. Since $\vartheta$ is an increasing mapping, we obtain $\vartheta(m) \leq \vartheta((m|n)|n)$ for each $m, n \in M$.

**Theorem 3.1.** *Let* $\vartheta : M \to M$ *be a Sheffer stroke very true operator. Let* sup *and* inf *be the least upper bound and greatest lower bound functions, respectively. Then the following equalities*

$$\sup\{\vartheta(m), \vartheta(n)\} = \vartheta(\sup\{m, n\}) \qquad \textit{and} \qquad \inf\{\vartheta(m), \vartheta(n)\} = \vartheta(\inf\{m, n\})$$

*are satisfied for each* $m, n \in M$.

**Proof.** Let $m, n \in M$ and the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. We have $m \leq \sup\{m, n\}$ and $n \leq \sup\{m, n\}$. Since $\vartheta$ is an increasing mapping, we get $\vartheta(n) \leq \vartheta(\sup\{m, n\})$ and $\vartheta(m) \leq \vartheta(\sup\{m, n\})$. Then, we obtain the following inequality

$$\sup\{\vartheta(m), \vartheta(n)\} \leq \vartheta(\sup\{m, n\}) \tag{3.3}$$

for each $m, n \in M$.

Let $\sup\{\vartheta(m), \vartheta(n)\} = k$ for $k \in M$. So, we have $\vartheta(m) \leq k$ and $\vartheta(n) \leq k$. By the help of Lemma 3.1 and Proposition 3.1 *(iii)*, we get $\vartheta(m) \leq \vartheta(k)$ and $\vartheta(n) \leq \vartheta(k)$. Using again Proposition 3.1 *(iii)*, we get $m \leq k$ and $n \leq k$. Then, we attain $\sup\{m, n\} \leq k$. From Definition 3.1 $(SV_{SM}2)$ and Proposition 3.1 *(iii)*, we obtain following the inequalities

$$\vartheta(\sup\{m, n\}) \leq \vartheta(k) \leq k = \sup\{\vartheta(m), \vartheta(n)\}. \tag{3.4}$$

From Inequalities (3.3) and (3.4), we prove that $\sup\{\vartheta(m), \vartheta(n)\} = \vartheta(\sup\{m, n\})$ for each $m, n \in M$.

For the infimum part of the proof, we have $\inf\{m, n\} \leq m$ and $\inf\{m, n\} \leq n$ for each $m, n \in M$. Since $\vartheta$ is an increasing mapping, we get $\vartheta(\inf\{m, n\}) \leq \vartheta(m)$ and $\vartheta(\inf\{m, n\}) \leq \vartheta(n)$. So, we obtain the following inequality

$$\vartheta(\inf\{m, n\}) \leq \inf\{\vartheta(m), \vartheta(n)\}. \tag{3.5}$$

By Definition 3.1 ($SV_{SM}2$), we have $\vartheta(m) \leq m$ and $\vartheta(n) \leq n$. Then, we get $\inf\{\vartheta(m), \vartheta(n)\} \leq \inf\{m, n\}$. From Proposition 3.1 (*iii*) and Lemma 3.1, we handle $\vartheta(\inf\{\vartheta(m), \vartheta(n)\}) \leq \vartheta(\vartheta(\inf\{m, n\}))$, i.e.,

$$\inf\{\vartheta(m), \vartheta(n)\} \leq \vartheta(\inf\{m, n\}). \tag{3.6}$$

From Inequalities (3.5) and (3.6), we show that $\inf\{\vartheta(m), \vartheta(n)\} = \vartheta(\inf\{m, n\})$ for each $m, n \in M$.

**Example 3.2.** *Let $M = \{0, k, l, m, n, 1\}$ and $\vartheta : M \to M$ be defined as Example 3.1. Then we show that Theorem 3.1 is satisfied for each $a, b \in M$. If one of $\{a, b\}$ equals 0 or 1, the equalities $\sup\{\vartheta(a), \vartheta(b)\} = \vartheta(\sup\{a, b\})$ and $\inf\{\vartheta(a), \vartheta(b)\} = \vartheta(\inf\{a, b\})$ are obtained clearly. We examine $a \in \{k, l, m, n\}$ and $b \in \{k, l, m, n\}$. So, we need to examine the sets such as $\{k, l\}$, $\{k, m\}$, $\{k, n\}$, $\{l, m\}$, $\{l, n\}$ and $\{m, n\}$.*

- *We analyze for $\{k, l\}$:*

$$\sup\{\vartheta(k), \vartheta(l)\} = \sup\{m, n\} = 1 = \vartheta(1) = \vartheta(\sup\{k, l\}).$$

$$\inf\{\vartheta(k), \vartheta(l)\} = \inf\{m, n\} = 0 = \vartheta(0) = \vartheta(\inf\{k, l\}).$$

- *We analyze for $\{k, m\}$:*

$$\sup\{\vartheta(k), \vartheta(m)\} = \sup\{m, m\} = m = \vartheta(m) = \vartheta(\sup\{k, m\}).$$

$$\inf\{\vartheta(k), \vartheta(m)\} = \inf\{m, m\} = m = \vartheta(k) = \vartheta(\inf\{k, m\}).$$

- *We analyze for $\{k, n\}$:*

$$\sup\{\vartheta(k), \vartheta(n)\} = \sup\{m, n\} = 1 = \vartheta(1) = \vartheta(\sup\{k, n\}).$$

$$\inf\{\vartheta(k), \vartheta(n)\} = \inf\{m, n\} = 0 = \vartheta(0) = \vartheta(\inf\{k, n\}).$$

- *We analyze for $\{l, m\}$:*

$$\sup\{\vartheta(l), \vartheta(m)\} = \sup\{n, m\} = 1 = \vartheta(1) = \vartheta(\sup\{l, m\}).$$

$$\inf\{\vartheta(l),\vartheta(m)\} = \inf\{n,m\} = 0 = \vartheta(0) = \vartheta(\inf\{l,m\}).$$

- *We analyze for* $\{l,n\}$*:*

$$\sup\{\vartheta(l),\vartheta(n)\} = \sup\{n,n\} = n = \vartheta(n) = \vartheta(\sup\{l,n\}).$$

$$\inf\{\vartheta(l),\vartheta(n)\} = \inf\{n,n\} = n = \vartheta(l) = \vartheta(\inf\{l,n\}).$$

- *We analyze for* $\{m,n\}$*:*

$$\sup\{\vartheta(m),\vartheta(n)\} = \sup\{m,n\} = 1 = \vartheta(1) = \vartheta(\sup\{m,n\}).$$

$$\inf\{\vartheta(m),\vartheta(n)\} = \inf\{m,n\} = 0 = \vartheta(o) = \vartheta(\inf\{m,n\}).$$

**Corollary 3.1.** *Let* $m,n \in M$ *and the mapping* $\vartheta : M \to M$ *be a Sheffer stroke very true operator. Then the following equalities*

$$\sup\{\vartheta(m),\vartheta(n)\} = \vartheta(\sup\{\vartheta(m),\vartheta(n)\}) \quad \textit{and} \quad \inf\{\vartheta(m),\vartheta(n)\} = \vartheta(\inf\{\vartheta(m),\vartheta(n)\})$$

*are verified for each* $m,n \in M$*.*

**Proof.** It is straightforward from Theorem 3.1 and Proposition 3.1 $(iii)$.

**Theorem 3.2.** *Let* $Fix_\vartheta(M)$ *be the set of the points of* $M$ *such that* $\vartheta(m) = m$*. Then, the equality* $Fix_\vartheta(M) = \vartheta(M)$ *is satisfied.*

**Proof.** Assume that $n \in \vartheta(M)$. Then, we have any element $m$ of $M$ such that $\vartheta(m) = n$. Using Lemma 3.1, we obtain $\vartheta(n) = \vartheta(\vartheta(m)) = \vartheta(m) = n$. So, we get $n \in Fix_\vartheta(M)$. Hence, we handle the following relation

$$\vartheta(M) \subseteq Fix_\vartheta(M). \tag{3.7}$$

Let $n \in Fix_\vartheta(M)$. This means that $\vartheta(n) = n$. Since $n \in M$, $n = \vartheta(n) \in \vartheta(M)$. Therefore, we get the following relation

$$Fix_\vartheta(M) \subseteq \vartheta(M). \tag{3.8}$$

From the relations (3.7) and (3.8), we prove that $Fix_\vartheta(M) = \vartheta(M)$.

**Example 3.3.** *Let* $M = \{0,k,l,m,n,1\}$ *and* $\vartheta : M \to M$ *be defined as Example 3.1. Then, we have* $Fix_\vartheta(M) = \{0,n,m,1\}$ *and also* $\vartheta(M) = \{0,n,m,1\}$*. So, we verify* $Fix_\vartheta(M) = \vartheta(M)$ *for Example 3.1.*

Now, when we consider on Theorem 3.1 and Theorem 3.2, we can reach the following corollary.

**Corollary 3.2.** *Let the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then the following equalities*

$$\sup\{Fix_{\vartheta}(M)\} = \vartheta(\sup(M)) \qquad \textit{and} \qquad \inf\{Fix_{\vartheta}(M)\} = \vartheta(\inf(M))$$

*are verified.*

**Lemma 3.4.** *Let $id : M \to M$ be defined as $Id(m) = m$ for each $m \in M$. Then, the mapping $Id$ is a Sheffer stroke very true operator on $M$.*

**Proof.** It is clear from Definition 3.1, Definition 2.6 and Theorem 2.1.

**Theorem 3.3.** *Let $\mathcal{M} = (M; |)$ be a Sheffer stroke $MTL$-algebra and the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then,*

*(i) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a $BL$-algebra if and only if $\vartheta(\inf\{m, n\}) = \vartheta((((m|m)|n)|n)$ $|(((m|m)|n)|n))$ for each very true operator $\vartheta$ on $M$ and for each $m, n \in M$,*

*(ii) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a $MV$-algebra if and only if $\vartheta(\sup\{m, n\}) = \vartheta((m|(n|n)|(n|n)))$ for each very true operator $\vartheta$ on $M$ and for each $m, n \in M$,*

*(iii) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a Gödel algebra if and only if $\vartheta(\inf\{m, n\}) = (\vartheta(m)|\vartheta(n))|$ $(\vartheta(m)|\vartheta(n))$ for each very true operator $\vartheta$ on $M$ and for each $m, n \in M$,*

**Proof.** The proof is clear from Lemma 3.4 and Theorem (3.7) in [18].

## 4. Conclusion

In this paper, we define Sheffer stroke very true operator on MTL-algebras. We get some fundamental properties of this operator. We give some equalities and inequalities which are used in our construction. Then, we attain some relations among very true operator, supremum and infimum relations. Finally, we construct paths among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them. After this work, we will use this operator other algebraic structures. By this means, we want to obtain new paths among new algebraic structures.

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