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CHARACTERIZATIONS OF CONTACT CR-WARPED PRODUCTS OF NEARLY COSYMPLECTIC MANIFOLDS IN TERMS OF ENDOMORPHISMS

WAN AINUN MIOR OTHMAN, SAYYEDAH A. QASEM, AND CENAP OZEL*

ABSTRACT. The main objective of this paper is to characterize contact CR-warped product submanifolds of a nearly cosymplectic manifold in terms of endomorphisms T and F. We also obtain some neccessary and sufficient conditions for integarbility of distributions involve in the definition.

1. INTRODUCTION

For a submanifold M of an almost Hermitain $(\widetilde{M}, J, g)$, we decompose JU into tangential and normal components as $JU = TU + FU$, for any vector field U tangent to M. Many researchers including B.-Y. Chen described geometric properties of subamnifolds in terms of T and F [9]. Later, such characterizations were extended for warped products in almost Hermitian as well as almost contact settings in [1], [2], [3], [5], [9], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. In the present paper, we obtain some results on the characterization of contact CR-warped product submanifolds of a *nearly cosymplectic* manifold in terms of endomorphisms T and F.

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* Corresponding author

The paper is organized as follows: In Section 2, we review some preliminary formulas and definitions. Section 3 is devoted to the study of contact CR-submanifold of a nearly cosymplectic manifold. In Section 4, we prove some lemmas on contact CR-warped product submanifolds of a *nearly cosymplectic* manifold, and then prove our main theorems on the characterization of warped product submanifolds in terms of the endomorphisms T and F.

2. Preliminaries

A $(2n+1)-$dimensional manifold $(\widetilde{M}, g)$ is said to be an *almost contact metric manifold* if it admits an endomorphism φ of its tangent bundle $T\widetilde{M}$, a vector field ξ, called *structure vector field* and η, the dual $1-$form of ξ satisfying the following.

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta \circ \varphi = 0, \tag{2.1}$$

and

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \ \eta(U) = g(U, \xi), \tag{2.2}$$

for any U, V tangent to $\widetilde{M}$ [8]. An almost contact metric structure (φ, ξ, η) is said to be a normal if almost complex structure J on a product manifold $\widetilde{M} \times R$ given by

$$J(U, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(U)\frac{d}{dt}),$$

where f is a smooth function on $\widetilde{M} \times R$, has no torsion, i.e., J is integrable, the condition for normality in term of φ, η and ξ is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on $\widetilde{M}$, where $[\varphi, \varphi]$ is the *Nijenhius tensor* of φ. Finally, the second fundamental $2-$form Φ is defined by $\Phi(U, V) = g(U, \varphi V)$. An almost contact metric structure (φ, η, ξ) is said to be *cosymplectic* if it is normal and both Φ and η are closed. They *characterized* by $(\widetilde{\nabla}_U \varphi)Y = 0$ *and* $\widetilde{\nabla}_U \xi = 0$. An almost contact metric structure (φ, η, ξ) is said to be *nearly cosymplectic* if φ is killing, i.e., if

$$(\widetilde{\nabla}_U \varphi)U = 0 \text{ or equivalently } (\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 0, \tag{2.3}$$

for any U, V tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ is the connection of the metric g on $\widetilde{M}$. If we replace $U = \xi$, $V = \xi$ in (2.3), we find that $(\widetilde{\nabla}_\xi \varphi)\xi = 0$ which is implies that $\varphi \widetilde{\nabla}_\xi \xi = 0$. Now applying φ and using (2.1), we get, $\widetilde{\nabla}_\xi \xi = 0$. Since from *Gauss formula* finally, we get $\nabla_\xi \xi = 0$ and $h(\xi, \xi) = 0$. The structure is said to be a *closely cosymplectic*, if φ is killing and η closed.

Now let M be a submanifold of $\widetilde{M}$. We will denote by ∇, the induced Riemannian connection on M and g, is the Riemannian metric on $\widetilde{M}$ as well as the metric induced on

M. Let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangent to M and normal to M, respectively and $\nabla^{\perp}$ the induced connection on $T^{\perp}M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the *Gauss* and *Weingarten* formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.4}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.5}$$

for each $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V) \tag{2.6}$$

Now for any $U \in \Gamma(TM)$, we write

$$\varphi U = TU + FU, \tag{2.7}$$

where TU and FU are the tangential and normal components of φU, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\varphi N = tN + fN, \tag{2.8}$$

where tN (resp. fN) is the tangential (resp. normal) component of φN. From (2.2) and (2.7), it is easy to observe that

$$g(TU, V) = -g(U, TV), \tag{2.9}$$

for each $U, V \in \Gamma(TM)$. The covariant derivatives of the endomorphism φ, T and F are defined, respectively as

$$(\widetilde{\nabla}_U \varphi)V = \widetilde{\nabla}_U \varphi V - \varphi \widetilde{\nabla}_U V, \quad \forall U, V \in \Gamma(T\widetilde{M}) \tag{2.10}$$

$$(\widetilde{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \quad \forall U, V \in \Gamma(TM) \tag{2.11}$$

$$(\widetilde{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \quad \forall U, V \in \Gamma(TM). \tag{2.12}$$

From [18] we have the following proposition

Proposition 2.1. *On any nearly cosymplectic manifold ξ is a killing form*

From the statement of above proposition we have the equality $g(\widetilde{\nabla}_U \xi, U) = 0$ for any vector field U tangent to nearly cosymplectic $\widetilde{M}$. We denote the tangential and normal parts of $(\widetilde{\nabla}_U \varphi)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ such that

$$(\widetilde{\nabla}_U \varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.13}$$

for all U, V tangent to M. Making use of (2.2)-(2.12) in (2.13), we can easily obtain

$$\mathcal{P}_U V =(\widetilde{\nabla}_U T)V - A_{FV} U - th(U, V), \tag{2.14}$$

$$\mathcal{Q}_U V =(\widetilde{\nabla}_U F)V + h(U, TV) - fh(U, V). \tag{2.15}$$

Similarly for any $N \in \Gamma(T^{\perp}M)$, denoting the tangential and normal parts of $(\widetilde{\nabla}_U \varphi)N$ by $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ such that

$$(\widetilde{\nabla}_U \varphi)N = \mathcal{P}_U N + \mathcal{Q}_U N. \tag{2.16}$$

Making use (2.3), (2.7), (2.8) in (2.16), we obtain

$$\mathcal{P}_U N = (\widetilde{\nabla}_U t)N + TA_N U - A_{fN} U \tag{2.17}$$

$$\mathcal{Q}_U N = (\widetilde{\nabla}_U f)N + h(U, tN) + FA_N U, \tag{2.18}$$

for all $U \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$,

$$\left.\begin{aligned}
&(i)\ \mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W,\ (ii)\ \mathcal{Q}_{U+V} W = \mathcal{Q}_U W + \mathcal{Q}_V W,\\
&(iii)\ \mathcal{P}_U (W + Z) = \mathcal{P}_U W + \mathcal{P}_U Z,\\
&(iv)\ \mathcal{Q}_U (W + Z) = \mathcal{Q}_U W + \mathcal{Q}_U Z,\\
&(v)\ g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W),\ (vi)\ g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N),\\
&(vii)\ \mathcal{P}_U \varphi V + \mathcal{Q}_U \varphi V = -\varphi(\mathcal{P}_U V + \mathcal{Q}_U V).
\end{aligned}\right\} \tag{2.19}$$

In a nearly cosymplectic manifold $\widetilde{M}$, we have

$$(i)\ \mathcal{P}_U V + \mathcal{P}_V U = 0,\ (ii)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0, \tag{2.20}$$

for any $U, V \in \Gamma(T\widetilde{M})$.

3. Contact CR-submanifolds of a nearly cosymplectic manifold

Definition 3.1. *A submanifold M tangent to the structure vector filed ξ of an almost contact metric manifold $\widetilde{M}$ is said to be invariant if $\varphi(T_x M) \subseteq (T_x M)$ and anti-invariant if $\varphi(T_x M) \subseteq (T_x^{\perp} M)$ for each $x \in M$.*

Definition 3.2. *A submanifold M tangent to structure vector field ξ of an almost contact metric manifold $\widetilde{M}$ is said to be a contact CR-submanifold if there exist a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that*

(i) *$TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus < \xi >$, where $< \xi >$ is 1-dimensional distribution spanned by ξ,*
(ii) *the distribution $\mathcal{D}$ is invariant, i.e., $\varphi(\mathcal{D}) \subseteq \mathcal{D}$,*
(iii) *the distribution $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $\varphi\mathcal{D}^{\perp}) \subseteq (T^{\perp}M)$.*

If μ is an *invariant* subspace under φ of normal bundle $T^{\perp}M$. Then, in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = F\mathcal{D}^{\perp} \oplus \mu$. Let us denotes the orthogonal porojections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by B and C, respectively. Then for any $U \in \Gamma(TM)$, we have

$$U = BU + CU + \eta(U)\xi, \tag{3.21}$$

where $BU \in \Gamma(\mathcal{D})$ and $CU \in \Gamma(\mathcal{D}^{\perp})$. From (2.7), (2.8) and (3.21), we have

$$TU = \varphi BU, \ FU = \varphi CU. \tag{3.22}$$

So we observe the following equalities

$$\left.\begin{array}{l} (i)\ TC = 0,\ (ii)\ FB = 0, \\ (iii)\ t(T^{\perp}M) \subseteq \mathcal{D}^{\perp}\ , (iv)\ f(T^{\perp}M) \subseteq \mu. \end{array}\right\} \tag{3.23}$$

Theorem 3.1. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}\oplus < \xi >$ is integrable if and only if*

$$2g(\nabla_X Y, Z) = g(h(Y, \varphi X), \varphi Z) + g(h(X, \varphi Y), \varphi Z), \tag{3.24}$$

for any $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, then we derive

$$\begin{aligned} g([X, Y], Z) &= g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_Y X, Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(\varphi\widetilde{\nabla}_Y X, \varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_Y \varphi X - (\widetilde{\nabla}_Y \varphi)X, \varphi Z). \end{aligned}$$

From (2.4) and (2.3), we get

$$\begin{aligned} g([X,Y],Z) =& g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g((\widetilde{\nabla}_X \varphi)Y,\varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g(\widetilde{\nabla}_X \varphi Y,\varphi Z) \\ &+ g(\varphi\widetilde{\nabla}_X Y,\varphi Z) \\ &= g(\widetilde{\nabla}_X Y, Z) - g(h(Y,\varphi X),\varphi Z) - g(h(X,\varphi Y),\varphi Z) \\ &+ g(\widetilde{\nabla}_X Y, Z) \\ &= 2g(\nabla_X Y, Z) - g(h(Y,\varphi X) + h(X,\varphi Y),\varphi Z). \end{aligned} \tag{3.25}$$

Our assertion follows from the above relation, which proves the theorem completely.

Lemma 3.1. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}\oplus < \xi >$ defines a totally geodesic foliation if and only if*

$$h(Y,\varphi X) + h(X,\varphi Y) \in \mu \tag{3.26}$$

for all $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$.

Proof. The distribution $\mathcal{D} \oplus \xi$ is a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(\mathcal{D} \oplus \xi)$ for any $X, Y \in \Gamma(\mathcal{D} \oplus \xi)$. Applying these definition in the Eq 3.25, we get the required proof.

Similarly, for anti-invariant distribution, we have

Theorem 3.2. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^{\perp}$ is integrable if and only if*

$$2g(\nabla_Z W, \varphi X) = g(h(X,Z),\varphi W) + g(h(X,W),\varphi Z) \tag{3.27}$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}\oplus < \xi >)$.

Proof. Let us derive

$$\begin{aligned} g([Z,W],\varphi X) &= g(\widetilde{\nabla}_Z W,\varphi X) - g(\widetilde{\nabla}_W Z,\varphi X) \\ &= g(\widetilde{\nabla}_Z W,\varphi X) + g(\varphi\widetilde{\nabla}_W Z, X) \\ &= g(\widetilde{\nabla}_Z W,\varphi X) + g(\widetilde{\nabla}_W \varphi Z, X) - g((\widetilde{\nabla}_W \varphi)Z, X), \end{aligned}$$

For any $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}\oplus < \xi >)$. From (2.4), (2.5) and (2.3), we obtain

$$\begin{aligned} g([Z,W],\varphi X) &= g(\nabla_Z W,\varphi X) - g(A_{\varphi Z}W, X) + g((\widetilde{\nabla}_Z \varphi)W, X) \\ &= g(\nabla_Z W,\varphi X) - g(A_{\varphi Z}W, X) + g(\widetilde{\nabla}_Z \varphi W, X) - g(\varphi\widetilde{\nabla}_Z W, X) \\ &= g(\nabla_Z W,\varphi X) - g(A_{\varphi Z}W, X) - g(A_{\varphi W}Z, X) + g(\nabla_Z W,\varphi X) \\ &= 2g(\nabla_Z W,\varphi X) - g(A_{\varphi Z}W, +A_{\varphi W}Z, X). \end{aligned} \tag{3.28}$$

Thus the desired result follows from the last the relation. It completes the proof of the theorem.

The following corollary is a consequence of the Theorem 3.2,

Corollary 3.1. *The anti-invariant distribution $\mathcal{D}^{\perp}$ of contact CR-submanifold M in a nearly cosymplectic manifold $\widetilde{M}$ is defines totally geodesic foliation if and only if*

$$A_{\varphi Z}W + A_{\varphi W}Z \in \Gamma(\mathcal{D}^{\perp}) \tag{3.29}$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$.

Proof. The proof follows from (3.28) and the definition of totally geodesic foliation.

Theorem 3.3. *The distribution $\mathcal{D}^{\perp}$ of a contact CR-submanifold M in a nearly cosymplectic manifold $\widetilde{M}$ is integrable if and only if*

$$g(\mathbf{P}_Z W,\varphi X) = 2\eta(X)g(\widehat{\nabla}_Z\xi, W)$$

or equivalent

$$g(A_{\varphi Z}W,\varphi X) = g(A_{\varphi W}Z,\varphi X), \tag{3.30}$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}\oplus < \xi >)$.

Proof. Let use the definition of Lie bracket, then simplification gives

$$g([Z,W], X) = g(\widetilde{\nabla}_Z W - \widetilde{\nabla}_W Z, X),$$

for $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}\oplus < \xi >)$. Using (2.2), we get

$$g([Z,W], X) = g(\varphi\widetilde{\nabla}_Z W - \varphi\widetilde{\nabla}_W Z,\varphi X) - \eta(X)g(\widehat{\nabla}_Z\xi, W) + \eta(X)g(\widehat{\nabla}_W\xi, Z).$$

Hence, using the property of covariant derivative (2.10), structure equation of a nearly cosymplectic manifold (2.3) and Proposition 2.1, we obtain

$$g([Z,W],X) = g(2\mathcal{P}_Z W - \widetilde{\nabla}_W \varphi Z + \widetilde{\nabla}_Z \varphi W, \varphi X) - 2\eta(X) g(\widehat{\nabla}_Z \xi, W).$$

Now from Weingarten formula (2.5), we have

$$g([Z,W],X) = g(2\mathcal{P}_Z W, \varphi X) - g(A_{\varphi Z} W - A_{\varphi W} Z, \varphi X) - 2\eta(X) g(\widehat{\nabla}_Z \xi, W),$$

which proves the our assertion. It compete proof of the Theorem.

4. Contact CR-warped products of nearly cosymplectic manifolds

The warped product manifolds are the generalized version of Riemannian product manifolds. The notion of warped product manifold defined as follows:

Let (B, g_1) and (F, g_2) be two Riemannian manifolds and f, a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $B \times F = (B \times F, g)$, where $g = g_1 + f^2 g_2$. A warped product manifold M is said to be a trivial warped product if its warping function f is constant. A trivial warped product $B \times F$ is nothing but Riemannian product $B \times_f F$ where ${}_fF$ is the Riemannian manifold with Riemannian metric $f^2 g_F$ which is homothetic to the original metric g_F of F. Bishop and O'Neill [7] also obtained the following lemma which provides some basic formulas on warped product manifolds

Lemma 4.1. *Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$ then*

(i) $\nabla_X Y \in \Gamma(TB)$,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f) Z$,
(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W) \nabla \ln f$,

where $\nabla \ln f$ is gradient of the function $\ln f$ which is defined as $g(\nabla \ln f, X) = X \ln f$, for any $X \in \Gamma(TB)$. Moreover, ∇ and ∇' are the Levi-Civitas connection on B and F, respectively.

It follows from Lemma 4.1 that B is totally geodesic submanifold in M and F is totally umbilical submanifold in M. In this way, we investigate the characterization of non-trivial warped product submanifolds $M_T \times_f M_\perp$ of nearly cosymplectic manifolds in terms of T and F. In terms tensor fields we have following characterization results.

Theorem 4.1. [9] *A CR-submanifold M of a Kaehler manifold $\widetilde{M}$ is a CR-product if and only if T is parallel, i.e.,*

$$\widetilde{\nabla}T = 0.$$

Theorem 4.2. [12] *A proper contact CR-submanifold M of a Kaehler manifold $\widetilde{M}$ is locally CR-warped product if and only if T satisfies:*

$$(\widetilde{\nabla}_U T)V = (TBU\mu)CU + g(CU, CV)J\nabla\mu$$

any $U, V \in \Gamma(TM)$, where C and B are the projections on $\mathcal{D}^\perp$ and $\mathcal{D}$, respectively.

In the proceeding these study, we derive the following results which are very important for proving the characterization theorem.

Lemma 4.2. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then*

(i) $(\widetilde{\nabla}_X T)Z = 0$, *(ii)* $(\widetilde{\nabla}_Z T)X = (TX\ln f)Z$,

(iii) $(\widetilde{\nabla}_\xi T)X = T\nabla_X\xi$, *(iv)* $(\widetilde{\nabla}_U T)\xi = -T\nabla_{BU}\xi$,

(v) $(\widetilde{\nabla}_U T)Z = g(CU, Z)T\nabla\ln f$.

for all $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\perp)$ and $U \in \Gamma(TM)$.

Proof. First part directly follows from (2.11), Lemma 4.1(ii) and using the fact that $TZ = 0,\ \forall\ Z \in \Gamma(TM_\perp)$. For the second part, we find

$$\begin{aligned}(\widetilde{\nabla}_Z T)X &= \nabla_Z TX - T\nabla_Z X\\ &= (TX\ln f)Z - (X\ln f)TZ\\ &= (TX\ln f)Z,\end{aligned}$$

which is (ii). Similarly, to prove (iii), we have

$$(\widetilde{\nabla}_U T)Z = \nabla_U TZ - T\nabla_U Z. \tag{4.31}$$

Since $TZ = 0,\ \forall\ Z \in \Gamma(TM_\perp)$ and using (3.21) in (4.31), we obtain

$$(\widetilde{\nabla}_U T)Z = -T\{\nabla_{BU}Z + \nabla_{CU}Z + \eta(U)\nabla_\xi Z\}.$$

From Lemma 4.1(ii), we derive

$$(\widetilde{\nabla}_U T)Z = -(BU\ln f)TZ - T\nabla_{CU}Z - \eta(U)(\xi\ln f)TZ.$$

Since $\xi \ln f = 0$, funded by [18], then using Lemma 4.1(iii), it is easily obtain that

$$(\widetilde{\nabla}_U T)Z = -T\{\nabla'_{CU} Z - g(CU, Z)\nabla \ln f\}$$
$$= g(CU, Z)T\nabla \ln f.$$

Now, for any $X, Y \in \Gamma(TM_T)$, then from (2.14) and (2.20)(i), we get

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2th(X, Y). \tag{4.32}$$

By equation (2.11) and the fact that M_T is totally geodesic in M, it follows that $(\widetilde{\nabla}_X T)Y$ lies in M_T, thus left hand side in (4.32) completely lies in M_T. Therefore equating the tangential components along M_T in las equation, we get $th(X, Y) = 0$, which means that $h(X, Y) \in \Gamma(\mu)$. Then from (4.32), we find

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 0. \tag{4.33}$$

If, we set $Y = \xi$ in (4.33), we simplifies

$$(\widetilde{\nabla}_X T)\xi + (\widetilde{\nabla}_\xi T)X = 0$$
$$(\widetilde{\nabla}_\xi T)X = T\nabla_X \xi,$$

which gives the third result of the lemma. It completes proof of lemma.
First characterization theorem in terms of ∇T.

Theorem 4.3. *Let M be a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ with both invariant and anti-invariant distributions are integrable. Then M is locally a CR-warped product if and only if*

$$(\widetilde{\nabla}_U T)U = (TBU\lambda)CU + ||CU||^2 T\nabla\lambda, \tag{4.34}$$

or equivalently

$$(\widetilde{\nabla}_U T)V + (\widetilde{\nabla}_V T)U = (TBV\lambda)CU + (TBU\lambda)CV + 2g(CU, CV)T\nabla\lambda, \tag{4.35}$$

for each $U, V \in \Gamma(TM)$ and λ is a C^∞-function on M satisfying $Z\lambda = 0$, for each $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Assume that M be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then applying (3.21) in $(\widetilde{\nabla}_U T)U$, we derive

$$(\widetilde{\nabla}_U T)U = (\widetilde{\nabla}_U T)BU + (\widetilde{\nabla}_U T)CU + \eta(U)(\widetilde{\nabla}_U T)\xi.$$

Again applying (3.21) and using Lemma 4.2(iv), we get

$$\begin{aligned}(\widetilde{\nabla}_U T)U =&(\widetilde{\nabla}_{BU} T)BU + (\widetilde{\nabla}_{CU} T)BU + (\widetilde{\nabla}_U T)CU \\ &+ \eta(U)(\widetilde{\nabla}_\xi T)TU - \eta(U)T\nabla_{BU}\xi.\end{aligned}$$

As M_T is totally geodesic in M, then the first term of right side in the above equation is zero by using (2.3) and from the Lemma 4.2(ii), (iii), (v), we arrive at

$$(\widetilde{\nabla}_U T)U = (TBU\lambda)CU + ||CU||^2 T\nabla\lambda,$$

where $\lambda = \ln f$. Hence, we obtain desire result (4.34). Furthermore, the equation (4.35) can be easily derive by replacing U by $U + V$ in (4.34).

Conversely, suppose that M is a contact CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ such that condition (4.35) holds. Then choosing $X, Y \in \Gamma(\mathcal{D}\oplus <\xi>)$ and using the fact that $CX = CY = 0$ in (4.35), we get the following condition, i.e.,

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 0. \tag{4.36}$$

Thus, from (2.20)(i), for nearly cosymplectic $\widetilde{M}$,

$$\mathcal{P}_X Y + \mathcal{P}_Y X = 0; \tag{4.37}$$

From (4.36), (4.37) and (2.17), we can easily obtain the condition $th(X, Y) = 0$, which is implies that $h(X, Y) \in \mu$ for all $X, Y \in \Gamma(\mathcal{D}\oplus <\xi>)$. Then using the integrability of $\mathcal{D}\oplus <\xi>$ and Theorem 3.1, which indicate that $g(\nabla_X Y, Z) = 0$, for all $Z \in \Gamma(\mathcal{D}^\perp)$. This proves that $\mathcal{D}\oplus <\xi>$ is parallel and each of its leaves M_T is totally geodesic in M. Furthermore, using the fact $BZ = BW = 0$, we get

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = 2g(Z, W)T\nabla\lambda, \tag{4.38}$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp)$. From (2.11), we have

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = A_{FZ}W + A_{FW}Z + 2th(Z, W). \tag{4.39}$$

Thus by (4.38) and (4.39), it follows that

$$A_{FZ}W + A_{FW}Z + 2th(Z, W) = 2g(Z, W)P\nabla\lambda. \tag{4.40}$$

Taking the inner product in (4.40) with $X \in \Gamma(D\oplus <\xi>)$, we obtain

$$g(A_{FZ}W, X) + g(A_{FW}Z, X) + 2g(th(Z, W), X) = 2g(Z, W)g(T\nabla\lambda, X). \tag{4.41}$$

The second term of right hand side in (4.41) is zero from (3.23)(iii), that is

$$g(h(X,W),\varphi Z)+g(h(X,Z),\varphi W)=2g(Z,W)g(T\nabla\lambda,X). \tag{4.42}$$

From the hypothesis of theorem that we assumed the totally real distribution is integrable. Then necessary and sufficient condition for integrability of $\mathcal{D}^{\perp}$ from the Theorem 3.2 and using (4.42), it follows that

$$\begin{aligned} g(\nabla_Z W,\varphi X) &= g(Z,W)g(T\nabla\lambda,X) \\ &= -g(Z,W)g(\nabla\lambda,\varphi X). \end{aligned} \tag{4.43}$$

As $\mathcal{D}^{\perp}$ is assumed to be integrable, then the second fundamental form of the immersion of $M_{\perp}(leafof\mathcal{D}^{\perp})$ into M is denoted by $h^{\perp}$. Hence, in point view Gauss formula (2.4) in (4.43), i.e.,

$$g(h^{\perp}(Z,W),\varphi X)=-g(Z,W)g(\nabla\lambda,\varphi X),$$

which is implies that

$$h^{\perp}(Z,W)=-g(Z,W)\nabla\lambda.$$

It means that $M_{\perp}$ is totally umbilical in M with mean curvature vector $H^{\perp}=-\nabla\lambda$. Now we can easily prove that $H^{\perp}$ is parallel corresponding to the normal connection ∇' of $M_{\perp}$ in M, i.e., $Z(\lambda)=0$ for all $Z\in\Gamma(D^{\perp})$ and $\nabla_Y\nabla\lambda\in\Gamma(\mathcal{D}\oplus<\xi>)$. Hence, the leaves of $\mathcal{D}^{\perp}$ are extrinsic spheres in M. From result of [11], we conclude that M is a warped product submanifold. The proof is done.

Lemma 4.3. *Let $M=M_T\times_f M_{\perp}$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then*

(i) $g((\widetilde{\nabla}_X F)Y,\varphi W)=0$, *(ii)* $g((\widetilde{\nabla}_X F)Z,\varphi W)=0$,

(iii) $g((\widetilde{\nabla}_Z F)X,\varphi W)=-(X\ln f)g(Z,W)$, *(iv)* $g((\widetilde{\nabla}_\xi F)Z,\varphi W)=0$,

(v) $g((\widetilde{\nabla}_Z F)W',\varphi W)=g(\mathbf{Q}_Z W',\varphi W)$,

for any $X,Y\in\Gamma(TM_T)$ and $Z,W,W'\in\Gamma(TM_{\perp})$.

Proof. Let M be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\widetilde{M}$, then,

$$\begin{aligned} g((\widetilde{\nabla}_X F)Y,\varphi W) &= g(-F\nabla_X Y,\varphi W) \\ &= -g(\nabla_X Y,W). \end{aligned}$$

As M_T is totally geodesic in M, then from the above equation, we get (i). To the other parts, from (2.18), it is easily seen that

$$((\widetilde{\nabla}_X F)Z, \varphi W) = g((\mathcal{Q}_X Z + fh(X, Z), \varphi W). \tag{4.44}$$

Using nearly cosymplectic manifold (2.3), and the property (v),(vii) of (2.19) in equation (4.44), we obtain

$$((\widetilde{\nabla}_X F)Z, \varphi W) = g(\varphi X, \mathcal{P}_Z W).$$

Then integrability Theorem 3.3 of the distribution $\mathcal{D}^\perp$, gives

$$((\widetilde{\nabla}_X F)Z, \varphi W) = 2\eta(X)g(\widehat{\nabla}_Z \xi, W) = 2\eta(X)(\xi \ln f)g(Z, W),$$

which is the result (ii) of lemma. Again, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$, we obtain

$$((\widetilde{\nabla}_Z F)X, \varphi W) = -g(F\nabla_Z X, \varphi W).$$

From Lemma 4.1(ii), we obtain (iii) as follows

$$g((\widetilde{\nabla}_Z F)X, \varphi W) = -(X \ln f)g(Z, W).$$

Now to prove (v), from (2.18), we find that

$$g((\widetilde{\nabla}_Z F)W', \varphi W) = g(\mathcal{Q}_Z W', \varphi W).$$

Similarly, we obtain

$$\begin{aligned} g((\widetilde{\nabla}_\xi F)Z, \varphi W) &= g(\mathcal{Q}_\xi Z + fh(\xi, Z), \varphi W) \\ &= g(\mathcal{Q}_\xi Z, \varphi W). \end{aligned}$$

Using the property (2.19)(vi), we can derive

$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = g(\varphi\xi, \mathcal{P}_Z W)$$
$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = 0,$$

which is the last result. It completes proof of the lemma.

Similarly, the second characterization theorem in terms of ∇F.

Theorem 4.4. *Assume that M be a contact CR-submanifold in a nearly cosymplectic manifold $\widetilde{M}$ with anti-invariant and invariant distributions are integrable. Then the M is locally a CR-warped product if only if*

$$g((\widetilde{\nabla}_U F)U, \varphi W) = -(BU\lambda)g(CU, W) \tag{4.45}$$

or equivalently

$$g((\widetilde{\nabla}_U F)V + (\widetilde{\nabla}_V F)U, \varphi W) = -(BU\lambda)g(CV, W) - (BV\lambda)g(CU, W) \tag{4.46}$$

for each $U, V \in \Gamma(TM)$ *and* λ *is a* C^∞*-function on* M *satisfying* $Z\lambda = 0$ *for each* $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let M be a contact CR-warped product submanifold in a nearly cosymplectic manifold $\widetilde{M}$. The property (3.21) gives

$$\begin{aligned} g((\widetilde{\nabla}_U F)V, \varphi W) =& g((\widetilde{\nabla}_{BU} F)BV, \varphi W) + g((\widetilde{\nabla}_{CU} F)BV, \varphi W) \\ &+ \eta(U)g((\widetilde{\nabla}_\xi F)BV, \varphi W) + g((\widetilde{\nabla}_{BU} F)CV, \varphi W) \\ &+ g((\widetilde{\nabla}_{CU} F)CV, \varphi W) + \eta(U)g((\widetilde{\nabla}_\xi F)CV, \varphi W) \\ &+ \eta(V)g((\widetilde{\nabla}_U F)\xi, \varphi W). \end{aligned}$$

Using Lemma 4.3, we obtain

$$g((\widetilde{\nabla}_U F)V, \varphi W) = g(\mathcal{Q}_{CU} CV, \varphi W) - (BV\lambda)g(CU, W). \tag{4.47}$$

By the polarization identity, we get

$$g((\widetilde{\nabla}_V F)U, \varphi W) = g(\mathcal{Q}_{CV} CU, \varphi W) - (BU\mu)g(CV, W). \tag{4.48}$$

From (4.47), (4.48) and (2.20)(ii), we get required result (4.46) or in particular, if we replace $V = U$ in (4.46) and using the property of nearly cosymplectic structure, i.e., $\mathcal{Q}_U U = 0$, we get first desired result of the theorem.

Conversely, let us consider that M be a CR-submanifold of a nearly cosymplectic manifold $\widetilde{M}$ with the condition (4.46) holds. Then using the fact that $CX = CY = 0$, in (4.46), simplification gives

$$g((\widetilde{\nabla}_X F)Y + (\widetilde{\nabla}_Y F)X, \varphi W) = 0,$$

for each $X, Y \in \Gamma(\mathcal{D} \oplus <\xi>)$. Thus, from the relations (2.18) and (2.20)(ii), we derive

$$2g(fh(X, Y), \varphi W) - g(h(X, TY) + h(Y, TX), \varphi W) = 0.$$

From the hypothesis of theorem, i.e., the distribution $(\mathcal{D} \oplus <\xi>)$ is integrable, then from the Theorem 3.1 gives $g(\nabla_X Y, W) = 0$, for all $W \in \Gamma(\mathcal{D}^\perp)$ which is implies that $\nabla_X Y \in (\mathcal{D} \oplus <\xi>)$. It means that the invariant distribution $(\mathcal{D} \oplus <\xi>)$ is a totally geodesic in M, i.e., the leaves of $(\mathcal{D} \oplus <\xi>)$ in M are totally geodesic. Similarly, other part, we have

$$g((\widetilde{\nabla}_X F)Z + (\widetilde{\nabla}_Z F)X, \varphi W) = -(X\lambda)g(Z, W),$$

for any $Z \in \Gamma(\mathcal{D}^{\perp})$, $X \in \Gamma(\mathcal{D} \oplus <\xi>$ and from (4.46). Then relations (2.12) and (2.18), we derive

$$g(\mathcal{Q}_X Z, \varphi W) - g(F\nabla_Z X, \varphi W) = -(X\lambda)g(Z, W).$$

On the other hand, the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable by hypothesis of the theorem. Thus first term of left hand side identicaly zero by the Theorem 3.3, then the above equation takes the form

$$g(\nabla_Z W, X) = -g(Z, W)g(\nabla\lambda, X).$$

Let $M_{\perp}$ denote the leaves of $\mathcal{D}^{\perp}$. If h' denotes the second fundamental form of the immersion of $M_{\perp}$ into M, then by the Gauss formula (2.4), we can write as

$$g(h'(Z, W), X) = -g(Z, W)g(\nabla\lambda, X),$$

which means that

$$h'(Z, W) = -g(Z, W)\nabla\lambda.$$

It implies that $M_{\perp}$ is totally umbilical in M with mean curvature vector $H = -\nabla\lambda$. Now we shall prove that H is parallel corresponding to the normal connection $\mathcal{D}$ of $M_{\perp}$ in M. In similar way of the Theorem 4.3, this means that the leaves of $\mathcal{D}^{\perp}$ are extrinsic spheres in M. Then by result of [11], M is locally a warped product. It completes proof the theorem.

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Institute of Mathematical Science, Faculty of Sciences, University of Malaya
Email address: wanainun@um.edu.my

Institute of Mathematical Science, Faculty of Sciences, University of Malaya
Email address: $sayydah_sharabi@yahoo.com$

Department of Mathematics, King Abdul Aziz University Jeddah Saudi Arabia.
Email address: cenap.ozel@gmail.com