International Journal of Maps in Mathematics
Volume:1, Issue:1, (2018), Pages:50-67
www.journalmim.com

ON THE GEOMETRY OF CONFORMAL ANTI-INVARIANT $\xi^{\perp}-$SUBMERSIONS

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Abstract. Lee [Anti-invariant $\xi^{\perp}-$ Riemannian submersions from almost contact manifolds, Hacettepe Journal of Mathematics and Statistic, 42(3), (2013), 231-241.] defined and studied anti-invariant $\xi^{\perp}-$Riemannian submersions from almost contact manifolds. The main goal of this paper is to consider conformal anti-invariant $\xi^{\perp}-$submersions (it means the Reeb vector field ξ is a horizontal vector field) from almost contact metric manifolds onto Riemannian manifolds as a generalization of anti-invariant $\xi^{\perp}-$Riemannian submersions. More precisely, we obtain the geometries of the leaves of $\ker\pi_*$ and $(\ker\pi_*)^{\perp}$, including the integrability of the distributions, the geometry of foliations, some conditions related to totally geodesicness and harmonicty of the submersions. Finally, we show that there are certain product structures on the total space of a conformal anti-invariant $\xi^{\perp}-$submersion.

1. Introduction

In the 1960s, B. O'Neill [22] and A. Gray [16] independently studied the notion of Riemannian submersions between Riemannian manifolds. In [31], B. Watson defined almost Hermitian submersions, meaning submersions defined on the Riemannian submersions between almost Hermitian manifolds. The author showed that the Riemannian submersion is also an almost complex mapping and consequently the horizontal and vertical distributions

Received:2018-02-10 *Revised: 2018-03-15* *Accepted:2018-03-19*

2010 Mathematics Subject Classification: 53C15, 53C40.

Key words: Almost contact metric manifold, conformal submersion, anti-invariant $\xi^{\perp}-$Riemannian submersion, conformal anti-invariant $\xi^{\perp}-$submersion

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are invariant with respect to the tensor field of type $(1,1)$ of the total space.

In [28], B. Sahin defined anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. We refer to some papers ([5], [15], [24], [26], [29]) related to the notion and the book [30]. The notion of almost contact Riemannian submersions between almost contact metric manifolds initiated by Chinea in [10]. He obtained the differential geometric properties among total space, fibers and base spaces.

One the other hand, Fuglede [11] and Ishihara [17], as a generalization of Riemannian submersion, introduced independently horizontally conformal submersions (see also: [4], [13], [23]). Gudmundsson and Wood [14], as a generalization of holomorphic submersions, defined the notion of conformal holomorphic submersions and obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism (see also [7], [8] and [9]). Recently, the first author of that paper in [1] considered conformal anti-invariant submersions, meaning submersions defined on cosymplectic manifolds such that the vertical distribution is anti invariant with respect to the almost contact structure (see also: [18], [19]). In this paper, we consider conformal anti-invariant $\xi^{\perp}-$submersions from an almost contact metric manifold under the assumption that the fibers are anti-invariant with respect to the tensor field of type $(1,1)$ of the almost contact manifold.

The paper is organized as follows. Section 2, we give some basic notions related to almost contact metric manifolds and conformal submersions. In third section, we introduce conformal anti-invariant $\xi^{\perp}-$submersions from almost contact metric manifolds onto Riemannian manifolds, and give main results for the geometry of a conformal anti-invariant $\xi^{\perp}-$submersion. The last section, we show that there are certain product structures on the total space of a conformal anti-invariant $\xi^{\perp}-$submersion.

2. Preliminaries

In this paper, all manifolds, vector fields and maps are assumed to be smooth unless otherwise stated.

2.1. **Almost contact metric manifolds.** Let (M, g_M) be an almost contact metric manifold with structure tensors (ϕ, ξ, η, g_M) where ϕ is a tensor field of type (1,1), ξ is a Reeb vector field, η is a 1-form and g_M is the Riemannian metric on M. Then these tensors satisfy [3]

$$\phi\xi = 0, \quad \eta o\phi = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad g_M(\phi X, \phi Y) = g_M(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

where I denotes the identity endomorphism of TM and X, Y are any vector fields on M. Moreover, if M is Sasakian [27], then we have

$$(\nabla_X \phi)Y = -g_M(X, Y)\xi + \eta(Y)X \quad \text{and} \quad \nabla_X \xi = \phi X, \tag{2.3}$$

where ∇ is the connection of Levi-Civita covariant differentiation.

2.2. **Conformal submersions.** Let $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a smooth map between Riemannian manifolds, and let $q \in M$. Then φ is called horizontally weakly conformal or semi conformal at q [4] if either (i) $d\varphi_q = 0$, or (ii) $d\varphi_q$ maps horizontal space $\mathcal{H}_q = (ker(d\varphi_q))^{\perp}$ conformally onto $T_{\varphi_*}N$, i.e., $d\varphi_q$ is surjective and there exists a number $\Lambda(q) \neq 0$ such that

$$g_N(d\varphi_q X, d\varphi_q Y) = \Lambda(q) g_M(X, Y) \ (X, Y \in \mathcal{H}_q).$$

We call the point q is of type (i) as a critical point if it satisfies the type (i), and we shall call the point q a regular point if it satisfied the type (ii). At a critical point, $d\varphi_q$ has rank 0; at a regular point, $d\varphi_q$ has rank n and φ is submersion. Also, the number $\Lambda(q)$ is called the *square dilation* (of φ at q). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M and it has no critical point, then we call it a (*horizontally conformal submersion*).

A vector field $Z \in \Gamma(TM)$ is called a basic vector field if $Z \in \Gamma((ker\pi_*)^{\perp})$ and $\pi-$related with a vector field $\bar{Z} \in \Gamma(TN)$ which means that $(\pi_{*q} Z_q) = \bar{Z}(\pi(q)) \in \Gamma(TN)$ for any $q \in \Gamma(TM)$.

O'Neill's tensors T and A defined for any $E, F \in \Gamma(TM)$ as follows;

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.4}$$

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.5}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [12]). And also, by using (2.4) and (2.5), for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$, we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W \tag{2.6}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + T_V X \tag{2.7}$$

$$\nabla_X V = A_X V + \mathcal{V}\nabla_X V \tag{2.8}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y \tag{2.9}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = A_X V$. Then it is well-know that

$$-g(A_X E, F) = g(E, A_X F) \text{ and } -g(T_V E, F) = g(E, T_V F)$$

for all $E, F \in T_x M$. T is exactly the second fundamental form of the fibres of π. For the special case when π is horizontally conformal we have the following:

Proposition 2.1. *([13]) Let $\pi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{\mathcal{V}[X, Y] - \lambda^2 g(X, Y) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.10}$$

Definition 2.1. *Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\pi : M \longrightarrow N$ is a smooth map between them. The second fundamental form of π is given by*

$$(\nabla\pi_*)(X, Y) = \nabla^{\pi}_X \pi_*(Y) - \pi_*(\nabla^M_X Y) \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$, where ∇^{π} is the pullback connection. It is obvious that the second fundamental form $(\nabla\pi_*)$ is symmetric.

Lemma 2.1. [32] *Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth map between them. Then we have*

$$\nabla^{\varphi}_X \varphi_*(Y) - \nabla^{\varphi}_Y \varphi_*(X) - \varphi_*([X, Y]) = 0$$

for $X, Y \in \Gamma(TM)$.

Remark 2.1. *From Lemma 2.1, for any X is basic vector field and $Y \in \Gamma(ker\pi_*)$, we obtain $[X, V] \in \Gamma(ker\pi_*)$. So, in this paper we assume that all horizontal vector fields are basic vector fields.*

Recall that π is called harmonic if the tension field $\tau(\pi) = trace(\nabla\pi_*) = 0$. (for details, see [4]).

Lemma 2.2. [4] *Let $\pi : M \longrightarrow N$ be a horizontally conformal submersion. Then, we have*

(a) $(\nabla\pi_*)(X, Y) = X(\ln\lambda)\pi_* Y + Y(\ln\lambda)\pi_* X - g(X, Y)\pi_*(\nabla\ln\lambda)$;

(b) $(\nabla\pi_*)(V, W) = -\pi_*(T_V W)$;

(c) $(\nabla\pi_*)(X, V) = -\pi_*(\nabla^M_X V) = -\pi_*(A_X V)$

for any $V \in \Gamma(ker\pi_*)$ and $X, Y \in (ker\pi_*)^{\perp}$.

Finally, we will mention the following from [25].

Let g_1 be a Riemannian metric tensor on the manifold $N = M_1 \times M_2$ and assume that the canonical foliations $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ intersect perpendiculary everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds $\Longleftrightarrow$ $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ are totally geodesic foliations.

3. Conformal anti-invariant $\xi^{\perp}-$submersions

In this section, we first define conformal anti-invariant $\xi^{\perp}-$submersions from an almost contact metric manifold onto a Riemannian manifold and derive the integrability of distributions, the geometry of foliations, some conditions related to totally geodesicness and harmonicity of the map. First of all, we give the definition of the submersion as follows:

Definition 3.1. *Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) be a Riemannian manifold. We suppose that there exist a horizontally conformal submersion $\pi : M \longrightarrow N$ such that ξ is normal to $ker\pi_*$ and $ker\pi_*$ is anti-invariant with respect to ϕ, i.e., $\phi(ker\pi_*) \subset (ker\pi_*)^{\perp}$. Then we say that π is a conformal anti-invariant $\xi^{\perp}-$submersion.*

Assume that if $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ is a conformal anti-invariant $\xi^{\perp}-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then from Definition 3.1, we have $\phi(ker\pi_*)^{\perp} \cap ker\pi_* \neq 0$. We denote the complementary orthogonal distribution to $\phi(ker\pi_*)$ in $(ker\pi_*)^{\perp}$ by μ. Then we write

$$(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus \mu. \tag{3.12}$$

Here, μ is an invariant distribution of $(ker\pi_*)^{\perp}$, with respect to ϕ and contains ξ. Given $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{3.13}$$

where $\mathcal{B}X \in \Gamma(ker\pi_*)$ and $\mathcal{C}X \in \Gamma(\mu)$. On the other hand, since $\pi_*((ker\pi_*)^{\perp}) = TN$ and π is a conformal submersion, using (3.13) we obtain $\lambda^{-2} g_N(\pi_*\phi V, \pi_*\mathcal{C}X) = 0$ for any $X \in \Gamma((ker\pi_*)^{\perp})$ and $V \in \Gamma(ker\pi_*)$, which implies that

$$TN = \pi_*(\phi ker\pi_*) \oplus \pi_*(\mu). \tag{3.14}$$

Remark 3.1. *We note that every anti-invariant $\xi^{\perp}-$submersion from an almost contact manifold onto a Riemannian manifold is a conformal anti-invariant $\xi^{\perp}-$submersion with $\lambda = 1$ [20].*

Lemma 3.1. *Let π be a conformal anti-invariant $\xi^{\perp}$-submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$A_X\xi = -\mathcal{B}X, \tag{3.15}$$

$$T_V\xi = 0, \tag{3.16}$$

$$g_M(\mathcal{C}Y, \phi W) = 0, \tag{3.17}$$

$$g_M(\nabla^M_X \mathcal{C}Y, \phi W) = -g_M(\mathcal{C}Y, \phi A_X W) \tag{3.18}$$

for $Y, \xi, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. By using (2.3), (2.9) and (3.13) we have (3.15). Using (2.3) and (2.7) we get (3.16). Given $Y \in \Gamma((ker\pi_*)^{\perp})$, $W \in \Gamma(ker\pi_*)$ and using (2.2), we have

$$g_M(\mathcal{C}Y, \phi W) = g_M(\phi Y - \mathcal{B}Y, \phi W) = g_M(\phi Y, \phi W) = g_M(Y, W) + \eta(Y)\eta(W) = g_M(Y, W) = 0,$$

due to $\mathcal{B}Y \in \Gamma(ker\pi_*)$ and $\phi W, \xi \in \Gamma((ker\pi_*)^{\perp})$. Differentiating (3.17) with respect to X, we get

$$\begin{aligned}
g_M(\nabla^M_X \mathcal{C}Y, \phi W) &= -g_M(\mathcal{C}Y, \nabla^M_X \phi W) \\
&= -g_M(\mathcal{C}Y, (\nabla^M_X \phi)W) - g_M(\mathcal{C}Y, \phi(\nabla^M_X W)) \\
&= -g_M(\mathcal{C}Y, \phi(\nabla^M_X W)) \\
&= -g_M(\mathcal{C}Y, \phi A_X W) - g_M(\mathcal{C}Y, \phi \mathcal{V}\nabla^M_X W) \\
&= -g_M(\mathcal{C}Y, \phi A_X W)
\end{aligned}$$

due to $\phi\mathcal{V}\nabla^M_X W \in \Gamma(\phi ker\pi_*)$. One can easily obtain the others.

As we know the distribution $ker\pi_*$ is integrable, we only deal with the integrability of the distribution $(ker\pi_*)^{\perp}$ and the geometry of the distributions.

Theorem 3.1. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$–submersion. Then the following conditions are equivalent to each other;*

(a) *The distribution $(ker\pi_*)^{\perp}$ is integrable,*

(b) $\lambda^{-2} g_N(\nabla^N_{\pi_* Y}\pi_*\mathcal{C}X - \nabla^{\pi}_X \pi_*\mathcal{C}Y, \pi_*\phi W) = g_M(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y$
$- 2g_M(\mathcal{C}X, Y)\ln\lambda - \eta(Y)X + \eta(X)Y, \phi W)$

for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. In view of (2.2) and (2.3), we get

$$g_M(\nabla^M_X Y, W) = g_M(\nabla^M_X \phi Y, \phi W) - \eta(Y) g_M(X, \phi W) \tag{3.19}$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $W \in \Gamma(ker\pi_*)$. Then, using (3.13) and (3.19), we find

$$\begin{aligned} g_M([X,Y],W) &= g_M(\nabla^M_X \phi Y, \phi W) - g_M(\nabla^M_Y \phi X, \phi W) - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W) \\ &= g_M(\nabla^M_X \mathcal{B} Y, \phi W) + g_M(\nabla^M_X \mathcal{C} Y, \phi W) - g_M(\nabla^M_Y \mathcal{B} X, \phi W) - g_M(\nabla^M_Y \mathcal{C} X, \phi W) \\ &\quad - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Using the property of π and (2.8) we derive

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B} Y - A_Y \mathcal{B} X, \phi W) + \lambda^{-2} g_N(\pi_*(\nabla^M_X \mathcal{C} Y), \pi_* \phi W) \\ &\quad - \lambda^{-2} g_N(\pi_*(\nabla^M_Y \mathcal{C} X), \pi_* \phi W) - \eta(Y) g_M(X, \phi V) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Hence, from (2.11) and Lemma 2.2 we get

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B} Y - A_Y \mathcal{B} X, \phi W) - g_M(\mathcal{H} \nabla \ln \lambda, X) g_M(\mathcal{C} Y, \phi W) \\ &\quad - g_M(\mathcal{H} \nabla \ln \lambda, \mathcal{C} Y) g_M(X, \phi W) + g_M(X, \mathcal{C} Y) g_M(\mathcal{H} \nabla \ln \lambda, \phi W) \\ &\quad + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C} Y, \pi_* \phi W) + g_M(\mathcal{H} \nabla \ln \lambda, Y) g_M(\mathcal{C} X, \phi W) \\ &\quad + g_M(\mathcal{H} \nabla \ln \lambda, \mathcal{C} X) g_M(Y, \phi W) - g_M(Y, \mathcal{C} X) g_M(\mathcal{H} \nabla \ln \lambda, \phi W) \\ &\quad - \lambda^{-2} g_N(\nabla^N_{\pi_* Y} \pi_* \mathcal{C} X, \pi_* \phi W) - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Furthermore, by using (3.17), we obtain

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B} Y - A_Y \mathcal{B} X - \mathcal{C} Y(\ln \lambda) X + \mathcal{C} X(\ln \lambda) Y - 2 g_M(\mathcal{C} X, Y) \ln \lambda \\ &\quad - \eta(Y) X + \eta(X) Y, \phi W) - \lambda^{-2} g_N(\nabla^N_{\pi_* Y} \pi_* \mathcal{C} X - \nabla^N_{\pi_* X} \pi_* \mathcal{C} Y, \pi_* \phi W). \end{aligned}$$

which means that $(a) \Leftrightarrow (b)$.

Using the integrability of $(ker\pi_*)^\perp$, from Theorem 3.1, we deduce:

Theorem 3.2. *Let π be a conformal anti-invariant $\xi^\perp-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two assertions below imply the third;*

(a) *The distribution $(ker\pi_*)^\perp$ is integrable.*

(b) *The map π is horizontally homothetic submersion.*

(c) $g_N(\nabla^\pi_Y \pi_* \mathcal{C} X - \nabla^\pi_X \pi_* \mathcal{C} Y, \pi_* \phi W) = \lambda^2 g_M(A_X \mathcal{B} Y - A_Y \mathcal{B} X - \eta(Y) X + \eta(X) Y, \phi W)$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $W \in \Gamma(ker\pi_*)$.

Proof. By the proof of Theorem 3.1, for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, we have

$$g_M([X,Y],W) = g_M(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y - 2g_M(\mathcal{C}X,Y)\ln\lambda$$
$$- \eta(Y)X + \eta(X)Y, \phi W) - \lambda^{-2}g_N(\nabla^{\pi}_Y\pi_*\mathcal{C}X - \nabla^{\pi}_X\pi_*\mathcal{C}Y, \pi_*\phi W).$$

Now, if we have (a) and (c), then we obtain

$$- g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(X,\phi W) + g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}X)g_M(Y,\phi W) \tag{3.20}$$
$$- 2g_M(\mathcal{C}X,Y)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0.$$

Now, taking $Y = \phi W$ in (3.20) for $W \in \Gamma(ker\pi_*)$, using (2.2) and (3.17), we derive

$$g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}X)g_M(\phi W,\phi W)) = g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}X)\{g_M(W,W) - \eta(W)\eta(W)\}$$
$$= g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}X)g_M(W,W) = 0.$$

Hence λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = \mathcal{C}X$ in (3.20) for $X \in \Gamma(\mu)$ and using (3.17) we obtain

$$- g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}^2Y)g_M(X,\phi W) + g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}X)g_M(\mathcal{C}X,\phi W)$$
$$- 2g_M(\mathcal{C}X,\mathcal{C}X)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0.$$

Thus, we get $2g_M(\mathcal{C}X,\mathcal{C}X)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0$ which means that the dilation λ is a constant on $\Gamma(\phi ker\pi_*)$. One can easily obtain the others.

Remark 3.2. *We assume that $(ker\pi_*)^{\perp} = \phi ker\pi_* \oplus \{\xi\}$. Using (3.13) one can prove that $\mathcal{C}X = 0$.*

Hence we obtain,

Corollary 3.1. *Let $\pi : (M,\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*)\oplus < \xi >$. Then the following conditions are equivalent to each other;*

(a) *The distribution $(ker\pi_*)^{\perp}$ is integrable*
(b) $A_X\phi Y + \eta(X)Y = A_Y\phi X + \eta(Y)X$
(c) $(\nabla\pi_*)(X,\phi Y) + \eta(Y)\pi_*X = (\nabla\pi_*)(Y,\phi X) + \eta(X)\pi_*Y$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$.

For the geometry of the distribution $(ker\pi_*)^{\perp}$, we get:

Theorem 3.3. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion. Then the following assertions are equivalent to each other;*

(a) *$(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*

(b) $-\lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi W) = g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi W)$

for $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$, $W \in \Gamma(ker\pi_*)$ and by using (2.2), (2.8), (2.9), (3.12), (3.13) and (3.19), we have

$$g_M(\nabla^M_X Y, W) = g_M(A_X BY, \phi W) + g_M(\nabla^M_X \mathcal{C}Y, \phi W) - \eta(Y) g_M(X, \phi W).$$

Using the property of π, (2.11) and Lemma (2.2) we arrive at

$$\begin{aligned} g_M(\nabla^M_X Y, W) = {} & g_M(A_X BY, \phi W) - \lambda^{-2} g_M(\mathcal{H}\nabla \ln\lambda, X) g_N(\pi_* \mathcal{C}Y, \pi_* \phi W) \\ & - \lambda^{-2} g_M(\mathcal{H}\nabla \ln\lambda, \mathcal{C}Y) g_N(\pi_* X, \pi_* \phi W) \\ & + \lambda^{-2} g_M(X, \mathcal{C}Y) g_N(\pi_*(\mathcal{H}\nabla \ln\lambda, \pi_* \phi W) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi V) - \eta(Y) g_M(X, \phi W) \end{aligned}$$

and using Definition 3.1 and (3.17) we arrive at

$$\begin{aligned} g_M(\nabla^M_X Y, W) = {} & g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi W) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi W) \end{aligned}$$

which tells that $(i) \Leftrightarrow (ii)$.

From Theorem 3.3, we obtain

Theorem 3.4. *Let π be a conformal anti-invariant $\xi^{\perp}-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two assertions below imply the third;*

(a) *The distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*

(b) *The map π is a horizontally homothetic submersion.*

(c) $g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi W) = \lambda^2 g_M(-A_X \mathcal{B}Y + \eta(Y)X, \phi W)$

for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, by the proof of Theorem 3.3, we have

$$\begin{aligned} g_M(\nabla^M_X Y, W) &= g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi W) \\ &\quad + \lambda^{-2} g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_*\phi W). \end{aligned}$$

Now, if we have (a) and (c), then we obtain

$$-g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(X, \phi W) + g_M(\mathcal{H}\nabla\ln\lambda, \phi W)g_M(X, \mathcal{C}Y) = 0. \tag{3.21}$$

Now, taking $X = \mathcal{C}Y)$ in (3.21) and using (3.17), we get $g_M(\mathcal{H}\nabla\ln\lambda, \phi W)g_M(X, \mathcal{C}Y) = 0$. Hence, λ is a constant on $\Gamma(\phi ker\pi_*)$. On the other hand, taking $X = \phi W$ in (3.21) and using (3.17) we find

$$\begin{aligned} g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(\phi W, \phi W)) &= g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)\{g_M(W, W) - \eta(W)\eta(W)\} \\ &= g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(W, W) = 0 \end{aligned}$$

which means that λ is a constant on $\Gamma(\mu)$. One can easily obtain the other assertions.

From the above theorem, we have the following:

Corollary 3.2. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus < \xi >$. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, the following conditions are equivalent to each other;*

(i) *The distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*
(ii) $A_X \mathcal{B}Y = \eta(Y)X$
(iii) $(\nabla\pi_*)(X, \phi W) = -\eta(Y)\pi_* X$.

Now, we investigate the geometry of $ker\pi_*$.

Theorem 3.5. *Let π be a conformal anti-invariant $\xi^{\perp}$−submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then for any $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$ the following conditiones are equivalent to each other;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*
(b) $-\lambda^{-2} g_N(\nabla^N_{\phi_* W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) = g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V\mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X)$.

Proof. Given $V, W \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma((ker\pi_*)^\perp)$, since $g_M(W,\xi)=0$, by using (2.3) we get $g_M(\nabla^M_V W,\xi) = -g_M(W,\nabla^M_V\xi) = -g_M(W,\phi V)=0$. Thus we get

$$\begin{aligned} g_M(\nabla^M_V W, X) &= g_M(\phi\nabla^M_V W, \phi X) + \eta(\nabla^M_V W)\eta(X) \\ &= g_M(\phi\nabla^M_V \phi W, \phi X) \\ &= g_M(\nabla^M_V \phi W, \phi X) - g_M((\nabla^M_V \phi)W, \phi X). \end{aligned}$$

Using (2.3), (2.6) and (3.13) we have

$$g_M(\nabla^M_V W, X) = g_M(T_V\phi W, \mathcal{B}X) + g_M(\mathcal{H}\nabla^M_V \phi W, \mathcal{C}X).$$

Since ∇^M is a Levi-Civita connection and $[V,\phi W] \in \Gamma(ker\pi_*)$ we derive

$$g_M(\nabla^M_V W, X) = g_M(T_V\phi W, \mathcal{B}X) + g_M(\nabla^M_{\phi W} V, \mathcal{C}X).$$

Using (2.3), (2.9) and taking into account μ is invariant, we have

$$\begin{aligned} g_M(\nabla^M_V W, X) &= g_M(T_V\phi W, \mathcal{B}X) + g_M(\phi\nabla^M_{\phi W} V, \phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \\ &= g_M(T_V\phi W, \mathcal{B}X) + g_M(\nabla^M_{\phi W} \phi V, \phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X). \end{aligned}$$

Now, using (2.11) and Lemma 2.2 (i) and using the property of π, we obtain

$$\begin{aligned} g_M(\nabla^M_U V, X) = {} & g_M(T_V\phi W, \mathcal{B}X) + \lambda^{-2} g_M(\mathcal{H}\nabla\ln\lambda, \phi W) g_N(\pi_*\phi V, \pi_*\phi\mathcal{C}X) \\ & - \lambda^{-2} g_M(\mathcal{H}\nabla\ln\lambda, \phi V) g_N(\pi_*\phi W, \pi_*\phi\mathcal{C}X) \\ & + g_M(\phi W, \phi V)\lambda^{-2} g_N(\pi_*(\mathcal{H}\nabla\ln\lambda, \pi_*\phi\mathcal{C}X) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \end{aligned}$$

and from Definition 3.1 and (3.17), we have

$$\begin{aligned} g_M(\nabla^M_U V, X) = {} & g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V\mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) \end{aligned}$$

so that we get $(i) \Leftrightarrow (ii)$.

From the above theorem, we deduce:

Theorem 3.6. *Let $\pi : (M,\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant $\xi^\perp$-submersion. Then, for any $V,W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^\perp)$, any two conditions below imply the third;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*

(b) *The dilation λ is a constant on $\Gamma(\mu)$.*

(c) $-\lambda^{-2}g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) = g_M(T_V\phi W, \mathcal{B}X) + \eta(\nabla^M_{\phi W}V)\eta(\mathcal{C}X)$.

Proof. Given $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$, by the proof of Theorem (3.5) we have

$$g_M(\nabla^M_W V, X) = g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V\mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W}V)\eta(\mathcal{C}X) + \lambda^{-2}g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X).$$

Now, if we have (a) and (c), then we get $g_M(\phi W, \phi V)g_M(\mathcal{H}\nabla\ln\lambda, \phi\mathcal{C}X) = 0$, which means that the dilation λ is a constant on $\Gamma(\mu)$. One can easily obtain the others.

Also we have,

Corollary 3.3. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*)\oplus < \xi >$. Then the following assertiones are equivalent to each other;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*

(b) $T_V\phi W = 0$

for $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

We note that a differential map π between two Riemannian manifolds is called a totally geodesic map $\Longleftrightarrow$ $(\nabla\pi_*)(Z_1, Z_2) = 0$, for any $Z_1, Z_2 \in \Gamma(TM)$.

Theorem 3.7. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$–submersion. Then π is a totally geodesic map if*

$$-\nabla^N_{\pi_*X}\pi_*Y_2 = \pi_*(\phi(A_X\phi Y_1 + \mathcal{V}\nabla^M_X\mathcal{B}Y_2 + A_X\mathcal{C}Y_2) + \mathcal{C}(\mathcal{H}\nabla^M_X\phi Y_1 + A_X\mathcal{B}Y_2 + \mathcal{H}\nabla^M_X\mathcal{C}Y_2)) - \eta(Y_2)\pi_*X - \{g_M(X, \phi Y_1) + g_M(X, \mathcal{C}Y_2)\}\pi_*\xi \tag{3.22}$$

for $X \in \Gamma((ker\pi_*)^{\perp}), Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker\pi_*)$ and $Y_2 \in \Gamma((ker\pi_*)^{\perp})$.

Proof. Using (2.2) and (2.11) we have

$$(\nabla\pi_*)(X, Y) = \nabla^N_{\pi_*X}\pi_*Y + \pi_*(-\nabla^M_X Y) = \nabla^N_{\pi_*X}\pi_*Y + \pi_*(\phi\nabla_X\phi Y - g(X, \phi Y)\xi - \eta(Y)X)$$

for any $X \in \Gamma((ker\pi_*)^\perp), Y \in \Gamma(TM)$. Then by using (2.8), (2.9) and (3.13) we get

$$\begin{aligned}(\nabla\pi_*)(X,Y) = \nabla^N_{\pi_*X}\pi_*Y_2 + \pi_*(\phi A_X\phi Y_1 + \mathcal{BH}\nabla^M_X\phi Y_1 + \mathcal{CH}\nabla^M_X\phi Y_1 + \mathcal{B}A_X\mathcal{B}Y_2 \\ + \mathcal{C}A_X\mathcal{B}Y_2 + \phi\mathcal{V}\nabla^M_X\mathcal{B}Y_2 + \phi A_X\mathcal{C}Y_2 + \mathcal{BH}\nabla^M_X\mathcal{C}Y_2 + \mathcal{CH}\nabla^M_X\mathcal{C}Y_2) \\ - \eta(Y_2)\pi_*X - \{g_M(X,\phi Y_1) + g_M(X,\mathcal{C}Y_2)\}\pi_*\xi\end{aligned}$$

for any $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker\pi_*)$ and $Y_2 \in \Gamma((ker\pi_*)^\perp)$. Thus taking into account the vertical parts, we obtain

$$\begin{aligned}(\nabla\pi_*)(X,Y) = \nabla^N_{\pi_*X}\pi_*Y + \pi_*(\phi(A_X\phi Y_1 + \mathcal{V}\nabla^M_X\mathcal{B}Y_2 + A_X\mathcal{C}Y_2) \\ + \mathcal{C}(\mathcal{H}\nabla^M_X\phi Y_1 + A_X\mathcal{B}Y_2 + \mathcal{H}\nabla^M_X\mathcal{C}Y_2)) \\ - \eta(Y_2)\pi_*X - \{g_M(X,\phi Y_1) + g_M(X,\mathcal{C}Y_2)\}\pi_*\xi.\end{aligned}$$

Hence $(\nabla\pi_*)(X,Y) = 0 \Longleftrightarrow$ (3.22) is satisfied.

For the totally geodesicness of the map, we also get:

Theorem 3.8. *Let $\pi : (M,\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant $\xi^\perp$-submersion. π is a totally geodesic map if and only if*

(a) $T_U\phi V = 0$ *and* $\mathcal{H}\nabla^M_U\phi V \in \Gamma(\phi ker\pi_*)$,

(b) *The map π is a horizontally homotetic map,*

(c) $A_Z\phi V = 0$ *and* $\mathcal{H}\nabla^M_Z\phi V \in \Gamma(\phi ker\pi)$

for $X, Y, Z \in \Gamma((ker\pi_*)^\perp)$ and $U, V \in \Gamma(ker\pi_*)$.

Proof. For any $U, V \in \Gamma(ker\pi_*)$, by using (2.3) and (2.11) we have

$$\begin{aligned}(\nabla\pi_*)(U,V) &= \nabla^N_{\pi_*U}\pi_*V + \pi_*(-\nabla^M_U V) \\ &= \pi_*(\phi\nabla^M_U\phi V - g_M(U,\phi V)\xi - \eta(V)X) \\ &= \pi_*(\phi\nabla^M_U\phi V).\end{aligned}$$

Then from (2.6) and (2.7) we arrive at

$$(\nabla\pi_*)(U,V) = \pi_*(\phi T_U\phi V + \mathcal{CH}\nabla^M_U\phi V).$$

From above equation, $(\nabla\pi_*)(U,V) = 0 \Longleftrightarrow \pi_*(\phi T_U\phi V + \mathcal{CH}\nabla^M_U\phi V) = 0$. Since ϕ is non-singular, $T_U\phi V = 0$ and $\mathcal{H}\nabla^M_U\phi V \in \Gamma(\phi ker\pi_*)$. On the other hand, from Lemma 2.2 we derive

$$(\nabla\pi_*)(X,Y) = X(\ln\lambda)\pi_*Y + Y(\ln\lambda)\pi_*X - g_M(X,Y)\pi_*(\nabla\ln\lambda)$$

for any $X, Y \in \Gamma(\mu)$. It is obvious that if π is a horizontally homotetic map, it follows that $(\nabla\pi_*)(X,Y) = 0$. Conversely, if $(\nabla\pi_*)(X,Y) = 0$, taking $Y = \phi X$ in the above equation, we get

$$X(\ln\lambda)\pi_*\phi X + \phi X(\ln\lambda)\pi_* X = 0.$$

Taking inner product with $\pi_*\phi X$ at the above equation we obtain

$$g_M(\nabla\ln\lambda, X)\lambda^2 g_M(\phi X, \phi X) + g_M(\nabla\ln\lambda, \phi X)\lambda^2 g_M(X, \phi X) = 0. \tag{3.23}$$

From (3.23), λ is a constant on $\Gamma(\mu)$. On the other hand, for $U, V \in \Gamma(ker\pi_*)$, from Lemma 2.2 we have

$$(\nabla\pi_*)(\phi U, \phi V) = \phi U(\ln\lambda)\pi_*\phi V + \phi V(\ln\lambda)\pi_*\phi U - g_M(\phi U, \phi V)\pi_*(\nabla\ln\lambda).$$

Again if π is a horizontally homothetic map, then $(\nabla\pi_*)(\phi U, \phi V) = 0$. Conversely, if $(\nabla\pi_*)(\phi U, \phi V) = 0$, putting U instead of V in above equation, we derive

$$2\phi U(\ln\lambda)\pi_*(\phi U) - g_M(\phi U, \phi U)\pi_*(\nabla\ln\lambda) = 0. \tag{3.24}$$

Taking inner product with $\pi_*\phi U$ at (3.24) and since π is a conformal submersion, we have

$$g_M(\phi U, \phi U)\lambda^2 g_M(\nabla\ln\lambda, \phi U) = 0$$

which means that the dilation λ is a constant on $\Gamma(\phi ker\pi_*)$. Thus the dilation λ is a constant on $\Gamma((ker\pi_*)^{\perp})$. Now, for $Z \in \Gamma(\mu)$ and $V \in \Gamma(ker\pi_*)$, from (2.3) and (2.11) we have

$$(\nabla\pi_*)(Z, V) = \pi_*(\phi\nabla^M_Z\phi V).$$

In view of (2.8) and (2.9) we have

$$(\nabla\pi_*)(Z, V) = \pi_*(\phi A_Z\phi V + \mathcal{C}\mathcal{H}\nabla^M_Z\phi V).$$

Hence $(\nabla\pi_*)(Z,V) = 0 \Longleftrightarrow \pi_*(\phi A_Z\phi V + \mathcal{C}\mathcal{H}\nabla^M_Z\phi V) = 0$. Since ϕ is non-singular, $A_Z\phi V = 0$ and $\mathcal{H}\nabla^M_Z\phi V \in \Gamma(\phi ker\pi_*)$. Therefore, we obtain the proof.

Now, we give some conditions related to harmonicity of the submersion.

Theorem 3.9. *Let $\pi : (M^{2(m+n)+1}, \phi, \xi, \eta, g_M) \longrightarrow (N^{m+2n+1}, g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion. Then the tension field τ of π is*

$$\tau(\pi) = -m\pi_*(\mu^{ker\pi_*}) + (1-m-2n)\pi_*(\nabla\ln\lambda) \tag{3.25}$$

where $\mu^{ker\pi_*}$ is the mean curvature vector field of the distribution of $ker\pi_*$.

Proof. Let $\{e_1,...,e_m,\phi e_1,...,\phi e_m,\xi,\mu_1,...,\mu_n,\phi\mu_1,...,\phi\mu_n\}$ be orthonormal basis of $\Gamma(TM)$ such that $\{e_1,...,e_m\}$ be orthonormal basis of $\Gamma(ker\pi_*)$, $\{\phi e_1,...,\phi e_m\}$ be orthonormal basis of $\Gamma(\phi ker\pi_*)$ and $\{\xi,\mu_1,...,\mu_n,\phi\mu_1,...,\phi\mu_n\}$ be orthonormal basis of $\Gamma(\mu)$. Then the trace of second fundamental form (restriction to $ker\pi_*\times ker\pi_*$) is given by

$$trace^{ker\pi_*}\nabla\pi_* = \sum_{i=1}^{m}(\nabla\pi_*)(e_i,e_i).$$

Then using (2.11) we obtain

$$trace^{ker\pi_*}\nabla\pi_* = -m\pi_*(\mu^{ker\pi_*}) \tag{3.26}$$

and also, we have

$$trace^{(ker\pi_*)^{\perp}}\nabla\pi_* = \sum_{i=1}^{m}(\nabla\pi_*)(\phi e_i,\phi e_i) + \sum_{i=1}^{2n}(\nabla\pi_*)(\mu_i,\mu_i) + (\nabla\pi_*)(\xi,\xi).$$

From Lemma 2.2 we get

$$\begin{aligned} trace^{(ker\pi_*)^{\perp}}\nabla\pi_* &= \sum_{i=1}^{m}2g_M(\mathcal{H}\nabla\ln\lambda,\phi e_i)\pi_*\phi e_i - m\pi_*(\nabla\ln\lambda)\\ &+\sum_{i=1}^{2n}2g_M(\mathcal{H}\nabla\ln\lambda,\mu_i)\pi_*\mu_i - 2n\pi_*(\nabla\ln\lambda)\\ &+2\xi(\ln\lambda)\pi_*\xi - \pi_*(\nabla\ln\lambda).\end{aligned}$$

Since $\{\frac{1}{\lambda(p)}\pi_{*p}(\phi e_i), \frac{1}{\lambda(p)}\pi_{*p}(\mu_h), \frac{1}{\lambda(p)}\pi_{*p}\xi\}_{p\in M,\ 1\leq i\leq m, 1\leq h\leq n}$ is an orthonormal basis of $T_{\pi(p)}N$ and using the properties of π, we derive

$$\begin{aligned} trace^{(ker\pi_*)^{\perp}}\nabla\pi_* &= \sum_{i=1}^{m}2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\phi e_i)\frac{1}{\lambda}\pi_*\phi e_i - m\pi_*(\nabla\ln\lambda)\\ &+\sum_{i=1}^{2n}2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\mu_i)\frac{1}{\lambda}\pi_*\mu_i - 2n\pi_*(\nabla\ln\lambda)\\ &+2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\xi)\frac{1}{\lambda}\pi_*\xi - \pi_*(\nabla\ln\lambda)\\ &=(1-m-2n)\pi_*(\nabla\ln\lambda)\end{aligned} \tag{3.27}$$

Then proof follows from (3.26) and (3.27).

From the above theorem, we have

Theorem 3.10. *Let $\pi:(M^{2(m+n)+1},\phi,\xi,\eta,g_M)\longrightarrow(N^{m+2n+1},g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion. Then any two conditions below imply the third:*

(a) *The map π is harmonic*

(b) *The fibres are minimal*

(c) *The map π is a horizontally homothetic map.*

Also, we have,

Corollary 3.4. *Let $\pi : (M^{2(m+n)+1}, \phi, \xi, \eta, g_M) \longrightarrow (N^{m+2n+1}, g_N)$ be a conformal anti-invariant $\xi^{\perp}$–submersion. π is harmonic if and only if the fibres are minimal.*

Now, we give some decomposition theorems comes from Theorem 3.3 and Theorem 3.5 in the following:

Theorem 3.11. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion. Then M is a locally product manifold if*

$$-\lambda^{-2} g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi V) = g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi V)$$

and

$$-\lambda^{-2} g_N(\nabla^{\pi}_{\phi W} \pi_* \phi V, \pi_* \phi \mathcal{C}X) = g_M(\phi \mathcal{C}X(\ln\lambda)\phi V - T_V \mathcal{B}X, \phi V) + \eta(\nabla_{\phi W} V)\eta(\mathcal{C}X)$$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U, V \in \Gamma(ker\pi_*)$, where $M_{(\ker\pi_*)^{\perp}}$ and $M_{(\ker\pi_*)}$ are integral manifolds of the distributions $(\ker\pi_*)^{\perp}$ and $(\ker\pi_*)$. Conversely, if M is a locally product manifold of the form $M_{(\ker\pi_*)^{\perp}} \times M_{(\ker\pi_*)}$ then we have

$$\lambda^{-2} g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi V) = g_M(\mathcal{C}Y(\ln\lambda)X - g_M(X, \mathcal{C}Y)\ln\lambda + \eta(Y)X, \phi V)$$

and

$$-\lambda^{-2} g_N(\nabla^{\pi}_{\phi W} \pi_* \phi V, \pi_* \phi \mathcal{C}X) = g_M(\phi \mathcal{C}X(\ln\lambda)\phi V, \phi V) + \eta(\nabla_{\phi W} V)\eta(\mathcal{C}X).$$

From Corollary 3.2 and Corollary 3.3, we have

Theorem 3.12. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus <\xi>$. Then M is a locally product manifold if $A_X \mathcal{B}Y = \eta(Y)X$ and $T_V \phi W = 0$ for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$.*

Acknowledgement

The authors are grateful to the referee for his/her valuable comments and suggestions.

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